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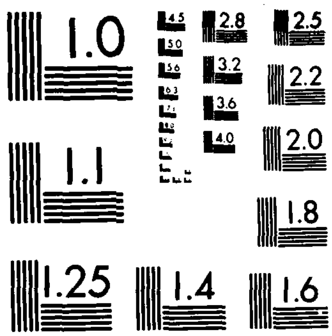

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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

STOCHASTIC EVOLUTION EQUATIONS WITH VALUES ON

THE DUAL OF A COUNTABLY HILBERT NUCLEAR SPACE

by

G. Kallianpur

and

V. Perez-Abreu

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THE DUAL OF A COUNTABLY HILBERT NUCLEAR SPACE

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Abstract

The work begins a study of stochastic evolution equations (SEE) driven by nuclear space valued martingales. The existence and uniqueness of solutions of perturbed SEE's is also considered. An illustration of the equations treated here is the SEE obtained by Mitoma in connection with the central limit theorem for the propagation of chaos.

Keywords: Nuclear spaces, stochastic evolution equations, Φ'-valued Wiener process, perturbation.

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Introduction and Assumptions

Let (Ω, F, P) be a complete probability space with a right continuous filtration $(F_t)_{t\geq 0}$ and let $(\Phi, \|\cdot\|_p \; p\geq 0)$ be a countably Hilbert nuclear space with Φ' its strong topological dual.

Consider the stochastic differential equation

$$\text{(I)} \qquad d\xi_t = A_t'\xi_t dt + P_t'\xi_t dt + dW_t$$

$$\xi_0 = \gamma$$

where: (Assumptions)

A1). - γ is a Φ'-valued F_0-measurable random variable such that for some $r_0 > 0$

$$E\|\gamma\|^2_{-r_0} < \infty.$$

A2). - $W = (W_t)_{t\geq 0}$ is a Φ'-valued Wiener process with covariance Q. This implies that there exists $q > 0$ such that

$$W_\bullet \in C(\mathbb{R}_+; \Phi_q') \text{ a.s.}$$

A3). - For each $t > 0$, $A_t : \Phi \to \Phi$ is a continuous linear operator that satisfies the following properties:

a). - The map $t \to A_t\phi$ is continuous on Φ for each $\phi \in \Phi$.

b). - $(A_t)_{t\geq 0}$ is the generator of a two parameter semigroup (evolution operator) $\{T(s,t) : 0 \leq s \leq t < \infty\}$ i.e.

$$T(s,t) = T(s,t')T(t',t) \qquad 0 \leq s < t' < t$$

$$T(t,t) = I,$$

$$\frac{d}{dt}T(s,t)\phi = T(s,t)A_t\phi \qquad \phi \in \Phi, \; s \leq t \text{ (Forward equation)},$$

$$\frac{d}{ds}T(s,t)\phi = -A_sT(s,t)\phi \qquad \phi \in \Phi, \; s \leq t \text{ (Backward equation)}.$$

and $T(s,t)$ satisfies the following conditions:

c). - For $s < t$ $T(s,t) : \Phi \to \Phi$ is a continuous linear operator.

d). - $\lim_{t \uparrow t_0} T(s,t)\phi = T(s,t_0)\phi$ in the Φ-topology for each s fixed and $0 \le s \le t_0$, $\phi \in \Phi$.

e). - $\lim_{s \uparrow s_0} T(s,t)\phi = T(s_0,t)\phi$ in the Φ-topology for each t fixed and $0 \le s_0 \le t$, $\phi \in \Phi$.

f). - For each $T > 0$ and $n \ge 0$

$$\sup_{0 \le s \le t \le T} \|T(s,t)\phi\|_n < \infty \quad \text{for all } \phi \in \Phi.$$

The next assumption concerns the perturbation operator P_t.

A4). - For each $t \ge 0$ $P_t : \Phi \to \Phi$ is a continuous linear operator on Φ and there exists a family of seminorms $\{|||\cdot|||_m : m \ge 0\}$ on Φ defining an equivalent topology on Φ to that given by the Hilbertian norms $\{\|\cdot\|_n : n \ge 0\}$ such that the following three conditions hold:

a). - For each $T > 0$ there exists $m_T > 0$ such that for each $m \ge m_T$ and $s \le T$, P_s has a continuous linear extension from $\Phi_{|m|}$ to $\Phi_{|m|}$ (denoted also by P_s), where $\Phi_{|m|}$ is the $|||\cdot|||_m$-completion of Φ and

b). - for each $\phi \in \Phi$ the map $s \to P_s\phi$ from $[0,T]$ to $\Phi_{|m|}$ is $\Phi_{|m|}$ continuous for $m \ge m_T$,

c). - $\sup_{0 \le s \le t \le T} |||P_s T(s,t)\phi|||_m \le K(m,T)\, |||\phi|||_m$ for all $\phi \in \Phi$ for $m \ge m_T$ and some constant $K(m,T) > 0$.

Remark 1. Condition A4(c) above can be obtained from A4(b) if we assume that for each $T > 0$ and $m \ge 0$

$$\sup_{0 \le s \le t \le T} |||T(s,t)\phi|||_m \le D(m,T)\, |||\phi|||_m \quad \text{for all } \phi \in \Phi$$

for some constant $D(m,T) > 0$.

Remark 2. Some authors, as Kato (1976) and Tanabe (1975), consider two parameter semigroups $T(s,t)$ on Banach or Hilbert spaces assuming that $T(s,t)$ is continuous on all the domain $\{(s,t) : 0 \leq s \leq t \leq T\}$. This is a stronger condition than A3(d)-(f).

In order to solve the SDE (I) we first consider the solution of the unperturbed SDE

$$d\eta_t = A_t'\eta_t dt + dW_t$$

$$\eta_0 = \gamma$$

for which it is possible to write a solution explicitly. This is done in Section 1 and is an extension of the work by Kallianpur and Wolpert (1984) and Christensen and Kallianpur (1985) who considered the case when $A_t = A$ $t \geq 0$ is the generator of a strongly continuous semigroup T_t. In Section 2 we solve the SDE

$$\xi_t = \int_0^t T(s,t)'P_s'\xi_s ds + \eta_t$$

and show that the solution of the above SDE is also a solution of (I). In Section 3 we extend the previous results to stochastic evolution equations with a nuclear space valued martingale as a driving term. Section 4 contains special cases and examples recently considered by Christensen and Kallianpur (1985), Hitsuda and Mitoma (1985) and Mitoma (1985). It is important to observe that the last two examples of Section 4 are instances where the two parameter evolution semigroup $T(s,t)$, its generator A_t and the perturbator P_t can all be defined directly on a countably Hilbertian nuclear space Φ so as to satisfy the above assumptions A1-A4. However, it is worth noting that, in many cases, these operators may be more naturally defined on a Hilbert or Banach space, as e.g., in the Example 4.1 or the works by Dawson and Gorostiza (1985), Kato (1976) and Tanabe (1975). In such cases the problem of finding a Φ for which the assumptions concerning A_t and P_t are valid, has to be solved first before our results can be applied.

1. Solution of the Unperturbed SDE

In this section we solve the SDE

$$\text{(II)} \qquad \begin{cases} d\xi_t = A_t'\xi_t dt + dW_t \\ \xi_0 = \gamma \end{cases}$$

where for each $t \geq 0$ $A_t' : \Phi' \to \Phi'$ is defined by the relation $(A_t'F)[\phi] = F[A\phi]$ for all $F \in \Phi'$, $\phi \in \Phi$.

Definition 1. We say that the SDE(II) has a Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ if the following four conditions hold:

a). - (ξ_t) is F_t-adapted and Φ'-valued.

b). - $\xi \in C(\mathbb{R}_+; \Phi')$ a.s.

c). - $\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + W_t[\phi]$ for all $\phi \in \Phi$ a.s. $t \geq 0$.

d). - For each $T > 0$

$$E(\sup_{0\leq t\leq T} |\xi_t[\phi]|^2) < \infty \quad \text{for all } \phi \in \Phi.$$

Proposition 1. If $\xi = (\xi_t)_{t\geq 0}$ is a solution of the SDE(II) then for each $T > 0$ there exists $n_T > 0$ and a version of ξ (also denoted by ξ) such that

$$\xi_\cdot^T \in C([0,T];\Phi'_{n_T}) \text{ a.s.}$$

and

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + W_t[\phi] \quad \text{for all } \phi \in \Phi,\ 0 \leq t \leq T \quad \text{a.s.}$$

Proof: Given $T > 0$ define

$$G_T^2(\phi) := E(\sup_{0\leq t\leq T} |\xi_t[\phi]|^2) < \infty.$$

Then by condition (d) in Definition 1 $G_T(\phi) < \infty$ for all $\phi \in \Phi$ and clearly G_T satisfies the conditions $G_T(\phi_1 + \phi_2) \leq G_T(\phi_1) + G_T(\phi_2)$ for $\phi_1, \phi_2 \in \Phi$ and $G_T(a\phi) = |a|G_T(\phi)$ $a \in \mathbb{R}$, $\phi \in \Phi$. Next since $\sup_{0\leq t\leq T}|\xi_t[\phi]|^2$ is a lower semicontinuous function of ϕ, by Fatou's Lemma $G_T(\phi)$ is also a lower semi-continuous function of ϕ. Then by a Baire category argument there exist $\theta_T > 0$ and $r_T > 0$ such that

$$E(\sup_{0\le t\le T} |\xi_t[\phi]|^2) \le \theta_T \|\phi\|^2_{r_T} \quad \text{for all } \phi \in \Phi.$$

Let $p_T > r_T$ such that the injection map $\Phi_{p_T} \hookrightarrow \Phi_{r_T}$ is Hilbert-Schmidt and let $\{\phi_j\}_{j\ge 1} \subset \Phi$ be a CONS for Φ_{p_T} with dual basis $\{\hat{\phi}_j\}_{j\ge 1}$ a CONS for Φ'_{p_T}. Then

$$E(\sum_{j=1}^{\infty} \sup_{0\le t\le T} |\xi_t[\phi_j]|^2) \le \theta_T \sum_{j=1}^{\infty} \|\phi_j\|^2_{r_T} < \infty.$$

Define

$$\Omega_T = \{\omega : \sum_{j=1}^{\infty} \sup_{0\le t\le T} |\xi_t(\omega)[\phi_j]|^2 < \infty\}$$

then $P(\Omega_T) = 1$. Next define

$$\hat{\xi}_t(\omega) = \begin{cases} \sum_{j=1}^{\infty} \xi_t(\omega)[\phi_j]\hat{\phi}_j & \omega \in \Omega_T \\ 0 & \omega \notin \Omega_T \end{cases}$$

Hence, $\hat{\xi}_t \in \Phi'_{p_T}$ a.s. and $\hat{\xi}_t(\omega)[\phi] = \xi_t(\omega)[\phi]$ for all $\phi \in \Phi$ $0 \le t \le T$ and $\omega \in \Omega_T$. Moreover by the dominated convergence theorem if $t, t_0 \in [0,T]$

$$\lim_{t\to t_0} \|\hat{\xi}_t(\omega) - \hat{\xi}_{t_0}(\omega)\|^2_{-p_T} = \lim_{t\to t_0} \sum_{j=1}^{\infty} (\xi_t(\omega)[\phi_j] - \xi_{t_0}(\omega)[\phi_j])^2$$

$$= \sum_{j=1}^{\infty} \lim_{t\to t_0} (\xi_t(\omega)[\phi_j] - \xi_{t_0}[\phi_j])^2 = 0.$$

Thus $\xi^T_\cdot \in C([0,T];\Phi'_{p_T})$ a.s. and therefore

$$P(\omega : N_T(\omega) : \sup_{0\le t\le T} \|\hat{\xi}_t\|^2_{-p_T} < \infty) = 1.$$

From now on we will write ξ_t instead of $\hat{\xi}_t$.

Next for $\omega \in \Omega_T$ and $0 \le t \le T$, for $\phi \in \Phi$ define

$$Y_t(\omega)[\phi] = \int_0^t \xi_s(\omega)[A_s\phi]ds.$$

We shall show that $Y_{\cdot}^T(\omega) \in C([0,T];\Phi_{m_T})$ for some $m_T > 0$. Suppressing ω in the writing we have that

$$|Y_t[\phi]| \le N_T \int_0^t \|A_s\phi\|_{p_T} ds$$

Then using the continuity of the map $s \to A_s\phi$ for all $\phi \in \Phi$, by a Baire category argument there exist $\theta_T' > 0$ and $m_T > p_T$ such that

$$\sup_{0\le t\le T} |Y_t[\phi]|^2 \le (N_T)^2\theta_T'\|\phi\|_{m_T}^2 \quad \text{for all } \phi \in \Phi.$$

Then $Y_t(\omega) \in \Phi_{m_T}'$ for all $0 \le t \le T$ $\omega \in \Omega_T$. Next let $\ell_T > m_T$ be such that the injection map $\Phi_{\ell_T} \hookrightarrow \Phi_{m_T}$ is Hilbert-Schmidt and let $\{e_j\}_{j\ge1} \subset \Phi$ be a CONS for Φ_{ℓ_T} with dual basis $\{\hat{e}_j\}_{j\ge1}$ a CONS for Φ_{ℓ_T}'. Then

$$\sum_{j=1}^{\infty} \sup_{0\le t\le T} |Y_t[e_j]|^2 \le (N_T\theta_T')^2 \sum_{j=1}^{\infty} \|e_j\|_{m_T}^2 < \infty .$$

Next from the inequality $|Y_t[\phi]| \le N_T\int_0^t\|A_s\phi\|_{p_T}ds$ we have that $Y_t[\phi]$ is a continuous function of t on $0 \le t \le T$ for each $\phi \in \Phi$. Then by the dominated convergence theorem

$$\lim_{t\to t_0} \|Y_t - Y_{t_0}\|_{-\ell_T}^2 = \lim_{t\to t_0} \sum_{j=1}^{\infty} (Y_t[e_j] - Y_{t_0}[e_j])^2 = 0 \quad t,t_0 \in [0,T]$$

i.e. $Y^T(\omega) \in C([0,T];\Phi_{\ell_T}')$ $\omega \in \Omega_T$.

Then we have shown that $\int_0^t A_s'\xi_s ds \in C([0,T];\Phi_{\ell_T}')$ a.s. for some $\ell_T > 0$. Hence taking $n_T = \max(r_0,q,p_T,\ell_T)$ we have that

$$Z_t = \gamma + \int_0^t A_s'\xi_s ds + W_t \in C([0,T];\Phi_{\ell_T}') \quad \text{a.s.}$$

Hence by conditions (b) and (c) in Definition 1 $P(Z_t = \xi_t \quad 0 \le t \le T) = 1$ and the proof of the proposition is complete. Q.E.D.

Remark 3. The following sufficient condition implies condition (d) in Definition 1: For each $T > 0$

$$E\int_0^T (\xi_s[A_s\phi])^2 ds < \infty \quad \text{for all } \phi \in \Phi.$$

Theorem 1. Under assumptions A1-A3 the SDE(II) has a unique Φ'-valued solution $\xi = (\xi_t)$ given by

(1.1) $$\xi_t = T'(0,t)\gamma + \int_0^t T'(s,t)A_s'W_s ds + W_t \quad t \ge 0$$

i.e.

(1.2) $$\xi_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t W_s[A_sT(s,t)\phi]ds + W_t[\phi] \quad \text{for all } \phi \in \Phi.$$

Furthermore, for each $T > 0$ there exists $\ell_T > 0$ such that

$$\xi_\cdot^T \in C([0,T];\Phi_{\ell_T}') \text{ a.s.}$$

and

$$E(\sup_{0\le t\le T} \|\xi_t\|_{-\ell_T}^2) < \infty$$

For the proof of Theorem 1 we will need the following two lemmas.

Lemma 1. For each $t \geq 0$ let $B_t : \Phi \to \Phi$ be a continuous linear operator and suppose that the map $t \to B_t\phi$ is continuous in the Φ-topology. Let $\{T(s,t) : 0 \leq s \leq t < \infty\}$ be a two parameter semigroup on Φ.

a). - Under assumption A3(c)-(e) the map $s \to B_s T(s,t)\phi$ is continuous in the Φ-topology for $0 \leq s \leq t < \infty$, $\phi \in \Phi$. Furthermore for $p \geq 0$ and $t > 0$

$$\sup_{0 \leq s \leq t} \|B_s T(s,t)\phi\|_p < \infty \quad \text{for all } \phi \in \Phi.$$

b). - If in addition we assume A3(f) then for each $p > 0$ and $T > 0$ there exist $r = r(B,T,p) > 0$ and $D = D(B,T,p) > 0$ such that

$$\sup_{0 \leq s \leq t \leq T} \|B_s T(s,t)\phi\|_p \leq D\|\phi\|_r \quad \text{for all } \phi \in \Phi.$$

Proof: a). - Since for each $t \geq 0$ $B_t : \phi \to \Phi$ is continuous then for each $p > 0$ the function $g_t(\phi) = \|B_t\phi\|_p$ is a continuous function on Φ and hence a lower semicontinuous function. Thus if $t \geq 0$

$$G_t(\phi) = \sup_{0 \leq s \leq t} \|B_s\phi\|_p \qquad \phi \in \Phi$$

is also a lower semicontinuous function. Moreover since the mapping $s \to B_s\phi$ is continuous then $G_t(\phi) < \infty$ for all $\phi \in \Phi$ and clearly $G_t(\phi_1 + \phi_2) \leq G_t(\phi_1) + (\phi_2)$, $G_t(a\phi_1) = |a|G_t(\phi_2)$ for $G \in \mathbb{R}$, $\phi_1, \phi_2 \in \Phi$. Then by a Baire category argument $G_t(\phi)$ is a continuous function of ϕ and there exist $\theta_t > 0$ and $r_t > 0$ such that

$$G(\phi) \leq \theta_t \|\phi\|_{r_t} \quad \text{for all } \phi \in \Phi.$$

Hence for each $s < t$ and $\phi \in \Phi$

$$\|B_s\phi\|_p \leq \theta_t \|\phi\|_{r_t} \quad \text{for all } \phi \in \Phi$$

and therefore for any $s_1 < t$ and $s_2 < t$

$$\|B_s(T(s_1,t)\phi - T(s_2,t)\phi)\|_p \leq \theta_t \|T(s_1,t)\phi - T(s_2,t)\phi\|_{r_t} \quad \text{for all } \phi \in \Phi.$$

Then if $s \uparrow s_0 \quad 0 \le s \le s_0 \le t$

$$\|B_s T(s,t)\phi - B_{s_0} T(s_0,t)\phi\|_p \le \|B_s(T(s,t)\phi - T(s_0,t)\phi)\|_p + \|B_s T(s_0,t)\phi - B_{s_0} T(s_0,t)\phi\|_p$$

$$\le \theta_t \|T(s,t)\phi - T(s_0,t)\phi\|_{r_t} + \|B_s T(s_0,t)\phi - B_{s_0} T(s_0,t)\phi\|_p$$

which goes to zero as $s \uparrow s_0$, the first term by assumption (A3)(e) and the second one since $s \to B_s \psi$ is a continuous mapping.

Hence the mapping $s \to B_s T(s,t)\phi$ is continuous in the Φ-topology on $0 \le s \le t < \infty$ and $\phi \in \Phi$ and therefore for $n \ge 0$ and $t \ge 0$

$$\sup_{0 \le s \le t} \|B_s T(s,t)\phi\| < \infty$$

which proves (a).

b). - From (a) we show that

$$G_T(\phi) = \sup_{0 \le t \le T} \|B_t \phi\|_p \le \theta_T \|\phi\|_{r_T} \quad \text{for all } \phi \in \Phi$$

i.e. $$\|B_s \phi\|_p \le \theta_T \|\phi\|_{r_T} \quad \text{for all } \phi \in \Phi \quad 0 \le t \le T.$$

Then for $0 \le s \le t \le T$

$$\|B_s T(s,t)\phi\|_p \le \theta_T \|T(s,t)\phi\|_{r_T} \quad \text{for all } \phi \in \Phi.$$

Next defining $V_T(\phi) = \sup_{0 \le s \le t \le T} \|T(s,t)\phi\|_{r_T}$ by A3(f) $V_T(\phi) < \infty$. Then since $V_T(\phi)$ is lower semicontinuous, $V_T(\phi_1 + \phi_2) \le V_T(\phi_1) + V_T(\phi_2)$ and $V_T(a\phi_1) = |a| V_T(\phi_1)$ $a \in \mathbb{R}$, $\phi_1, \phi_2 \in \Phi$, by a Baire category argument there exists $\theta'_T > 0$ and $r'_T > 0$ such that

$$V_T(\phi) \le \theta'_T \|\phi\|_{r'_T} \quad \text{for all } \phi \in \Phi$$

i.e.

$$\sup_{0 \le s \le t \le T} \|B_s T(s,t)\phi\|_p \le D_T \|\phi\|_{r'_T}$$

Q.E.D.

Lemma 2. Assume A3(a)-(f) and let B be a continuous linear operator from Φ to Φ. Then for each $F \in \Phi'$ and $0 \le u \le t$

a). $-F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BT(u,s)A_s\phi]ds$ for all $\phi \in \Phi$

b). $-F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BA_sT(s,t)\phi]ds$ for all $\phi \in \Phi$.

Proof: From A3(b)-(d) we have

$$\frac{d}{ds}T(u,s)\phi = \lim_{\varepsilon \downarrow 0} \frac{T(u,s+\varepsilon)\phi - T(u,s)\phi}{\varepsilon}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{T(u,s)T(s,s+\varepsilon)\phi - T(u,s)\phi}{\varepsilon} = T(u,s)A_s\phi$$

i.e.

$$\frac{d}{ds}T(u,s)\phi = T(u,s)A_s\phi \quad \phi \in \Phi \quad 0 \le u \le s < \infty.$$

Let $r_F > 0$ be such that $\|F\|_{-r_F} < \infty$. Then since $B : \Phi \to \Phi$ is continuous there exist $\theta = \theta_B > 0$ and $r = r_B > 0$ such that

$$\|B\psi\|_{r_F} \le \theta_B \|\psi\|_{r_B} \quad \text{for all } \psi \in \Phi.$$

Hence using the above inequality, Lemma 1(b) and A3 we have that for $T > 0$ and $0 \le u \le s \le T$

$$\|BT(u,s)A_s\phi\|_{r_F} \le \theta_B \|T(u,s)A_s\phi\|_{r_B} \le \theta_B D\|\phi\|_r \quad \text{for all } \phi \in \Phi$$

for some $r > 0$. Then

$$\sup_{0 \le u \le s \le T} |F[BT(u,s)A_s\phi]| \le \|F\|_{-r_F} \sup_{0 \le u \le s \le T} \|BT(u,s)A_s\phi\|_{r_F} < \infty \text{ for all } \phi \in \Phi$$

and $F[BT(u,s)A_s\phi]$ is integrable on $u \le s \le T$, $T > 0$.

Hence using the Forward equation, since F and B are continuous on Φ

$$\int_u^t F[BT(u,s)A_s\phi]ds = \int_u^t F[B\frac{d}{ds}T(u,s)\phi]ds$$

$$= \int_u^t \frac{d}{ds}F[BT(u,s)\phi]ds = F[BT(u,t)\phi] - F[BT(u,u)\phi]$$

i.e. $$F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BT(u,s)A_s\phi]ds$$

which proves (a).

b). - As in (a) we obtain the Backward equation

$$\frac{d}{ds}T(s,t)\phi = -A_sT(s,t)\phi$$

Taking $B_s = BA_s$ in Lemma 1(a) we have that

$$\sup_{0\le s\le t} ||BA_sT(s,t)\phi||_{r_F} < \infty$$

and hence as in (a) $|F[BA_sT(s,t)\phi]|$ is integrable on $0\le u\le s\le t$. Then using the Backward equation and the fact that B and F are continuous we obtain that

$$\int_u^t F[BA_sT(s,t)\phi]ds = -\int_u^t F[B\frac{d}{ds}T(s,t)\phi]ds = -\int_u^t \frac{d}{ds}F[BT(s,t)\phi]ds = -F[B\phi] + F[BT(u,t)\phi]$$

i.e.

$$F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BA_sT(s,t)\phi]ds.$$

Q.E.D.

Proof of Theorem 1. Let

$$\Omega_1 = \{\omega\in\Omega : W.(\omega)\in C(\mathbb{R}_+;\Phi'_q)\}\cap\{\omega : ||\gamma(\omega)||_{-r_0}<\infty\}$$

then by A1 and A2 $P(\Omega_1) = 1$.

Let $\omega\in\Omega_1$ (we will suppress ω when there is no conflict) and let $T > 0$.

Step 1. We shall prove that for each $0\le t\le T$ and $\omega\in\Omega_1$ the map

$$\phi \to Y_t(\omega)[\phi] = \int_0^t W_s(\omega)[A_sT(s,t)\phi]ds$$

is a continuous linear map, i.e. $Y_t(\omega)\in\Phi'$.

If we show that the integral is finite then clearly the map Y_t is linear. Define

$$K_t(\phi) = \int_0^t ||A_sT(s,t)\phi||_q ds \qquad \phi\in\Phi.$$

Then since A_s and $T(s,t)$ $0 \le s \le t \le T$ are continuous linear operators from Φ to Φ, if $\phi_n \to \phi$ in Φ

$$\|A_s T(s,t)\phi_n\|_m \underset{n\to\infty}{\to} \|A_s T(s,t)\phi\|_m \quad \text{all } m \ge 1.$$

Then by Fatou's Lemma $K_t(\phi)$ is a lower semicontinuous function on Φ and by Lemma 1(a) $K_t(\phi) < \infty$ for all $\phi \in \Phi$. Also $K_t(\phi_1 + \phi_2) \le K_t(\phi_1) + K_t(\phi_2)$, $K_t(a\phi_1) = |a| K_t(\phi_1)$ $a \in \mathbb{R}$, $\phi_1, \phi_2 \in \Phi$. Then by a Baire category argument there exist $\theta_t > 0$ and $r_t > 0$ such that

$$K_t(\phi) \le \theta_t \|\phi\|_{r_t} \quad \text{for all } \phi \in \Phi.$$

Thus

$$\left|\int_0^t W_s[A_s T(s,t)\phi]ds\right| \le \sup_{0\le s\le T} \|W_s\|_{-q} \int_0^t \|A_s T(s,t)\phi\|_q ds$$

$$\le \sup_{0\le s\le T} \|W_s\|_{-q}\theta_t \|\phi\|_{r_t} \quad \text{for all } \phi \in \Phi$$

and therefore $\int_0^t W_s[A_s T(t,s)\phi]ds$ is continuous and linear on Φ i.e.

$$\int_0^t T(s,t)'A_s'W_s(\omega)ds \in \Phi' \quad 0 \le t \le T.$$

Then from (1.1) $\xi_t(\omega) \in \Phi'$ for each $\omega \in \Omega_1$ and $t \ge 0$.

<u>Step 2</u>. We shall prove that $(\xi_t)_{t\ge 0}$ satisfies (c) in Definition 1, i.e. it must satisfy that for each $t \ge 0$ with probability one

$$(1.3) \qquad \xi_t[\phi] = \gamma[\phi] + W_t[\phi] + \int_0^t \xi_s[A_s\phi]ds \quad \text{for all } \phi \in \Phi$$

Applying Lemma 2(a) to $B = I$, $F = \gamma$ and $u = 0$ we have for all $\phi \in \Phi$

$$(1.4) \qquad \gamma[T(0,t)\phi] = \gamma[\phi] + \int_0^t \gamma[T(0,s)A_s\phi]ds.$$

Taking $F = W_u$ and $B = A_u$ in Lemma 2(a) we obtain

$$(1.5) \qquad W_u[A_u T(u,t)\phi] = W_u[A_u\phi] + \int_0^t W_u[A_u T(u,s)A_s\phi]ds.$$

Using (1.4) in (1.2) we have that for $\phi \in \Phi$

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \gamma[T(0,s)A_s\phi]ds + W_t[\phi] + \int_0^t W_u[A_uT(u,t)\phi]ds$$

and using (1.5) in the last term of the above expression and applying Fubini's Theorem we obtain that for all $\phi \in \Phi$

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \gamma[T(0,s)A_s\phi]ds + W_t[\phi] + \int_0^t \{W_u[A_u\phi] + \int_u^t W_u[A_uT(u,s)A_s\phi]ds\}du$$

$$= \gamma[A_0\phi] + W_t[\phi] + \int_0^t \gamma[T(0,s)A_s\phi]ds + \int_0^t W_s[A_s\phi]ds + \int_0^t (\int_0^s W_u[A_uT(u,s)A_s\phi]du)ds$$

$$= \gamma[\phi] + W_t[\phi] + \int_0^t \{\gamma\{T(0,s)A_s\phi] + W_s[A_s\phi] + \int_0^s W_u[A_uT(u,s)A_s\phi]du\}ds$$

$$= \gamma[\phi] + W_t[\phi] + \int_0^t \xi_s[A_s\phi]ds$$

i.e. $$\xi_t[\phi] = \gamma[\phi] + W_t[\phi] + \int_0^t \xi_s[A_s\phi]ds \quad 0 \le t \le T \quad \text{a.s.}$$

and therefore (1.2) satisfies (1.3).

Observe that $(t,\omega) \to \xi_t(\omega)$ is $\mathcal{B}(\Phi')/\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable and for each $t \ge 0$ ξ_t is $\mathcal{F}_t^{W,\gamma}$-measurable where

$$\mathcal{F}_t^{W,\gamma} = \sigma\{\gamma[\phi], W_s[\phi] : 0 \le s \le t, \ \phi \in \Phi\}.$$

Step 3. For a.a. ω P $t \to \xi_t(\omega)[\phi]$ is continuous. Let $\omega \in \Omega_1$. From (1.2) it is enough to show that

$$Y_t[\phi] = \int_0^t W_s[A_sT(s,t)\phi]ds$$

is continuous on t for each $\phi \in \Phi$. Let $T > 0$ and $0 \le t_0 \le t \le T$, then

$$(1.6) \qquad Y_t[\phi] - Y_{t_0}[\phi] = \int_0^t W_u[A_uT(u,t)\phi]du - \int_0^{t_0} W_u[A_uT(u,t_0)\phi]du$$

$$= \int_0^{t_0} \{W_u[A_uT(u,t)\phi] - W_u[A_uT(u,t_0)\phi]\}ds + \int_{t_0}^t W_u[A_uT(u,t)\phi]du.$$

Using Lemma 2(a) with $F = W_u$, $B = A_u$ we obtain

$$W_u[A_uT(u,t)\phi] = W_u[A_u\phi] + \int_u^t W_u[A_uT(u,s)A_s\phi]ds$$

and again applying Lemma 2(a) to $F = W_u$, $B = A_u$ and $t = t_0$

$$W_u[A_uT(u,t_0)\phi] = W_u[A_u\phi] + \int_u^{t_0} W_u[A_uT(u,s)A_s\phi]ds$$

and therefore

$$\{W_u[A_uT(u,t)\phi] - W_u[A_uT(u,t_0)\phi]\} = \int_{t_0}^{t} W_u[A_uT(u,s)A_s\phi]ds.$$

Using the last expression in (1.6) we have

$$Y_t[\phi] - Y_{t_0}[\phi] = \int_0^{t_0}\int_{t_0}^{t} W_u[A_uT(u,s)A_s\phi]dsdu + \int_{t_0}^{t} W_u[A_uT(u,t)\phi]du.$$

From Lemma 1(b) for some $r_1 = r_1(A,T,q) > 0$ and $D_1 = D_1(A,T,q) > 0$

$$\sup_{0\le u\le s\le T} ||A_uT(u,s)A_s\phi||_q \le D||\phi||_r \quad \text{for all } \phi \in \Phi.$$

Hence

$$|Y_t[\phi] - Y_{t_0}[\phi]| \le \int_0^{t_0}\int_{t_0}^{t} |W_u[T(u,s)A_s\phi]|dsdu + \int_{t_0}^{t} |W_u[A_uT(u,t)\phi]|du$$

$$\le \sup_{0\le s\le T} ||W_s||_{-q}\{t_0(t - t_0)D||\phi||_r + (t - t_0)D||\phi||_r\}$$

i.e. for $0 \le t_0 \le t \le T$

$$|Y_t[\phi] - Y_{t_0}[\phi]| \le \sup_{0\le s\le t} ||W_s||_{-q}TD||\phi||_r(t - t_0)$$

and similarly if $0 \le t \le t_0 \le T$

$$|Y_t[\phi] - Y_{t_0}[\phi]| \le \sup_{0\le s\le T} ||W_s||_{-q}TD||\phi||_r(t_0 - t)$$

i.e. for $\omega \in \Omega_1$ and $\phi \in \Phi$

$$|Y_t(\omega)[\phi] - Y_{t_0}(\omega)[\phi]| \le \sup_{0\le s\le T} \|W_s(\omega)\|_{-q} TD\|\phi\|_r |t-t_0| \qquad t,t_0 \in [0,T].$$

Hence $Y_t(\omega)[\phi]$ is continuous in t for all $\phi\in\Phi$ $0\le t\le T$ on a set of probability one. Moreover from the above expression we obtain

$$\text{(1.7)} \qquad \sup_{0\le t\le T} |Y_t(\omega)[\phi]| \le \sup_{0\le s\le T} \|W_s(\omega)\|_{-q} T^2 D\|\phi\|_r \quad \text{for all } \phi\in\Phi.$$

Also from the last expression and (1.2) we have that condition (d) in Definition 1 is satisfied.

<u>Step 4</u>. We shall prove that $\xi^T_{\cdot} \in C([0,T];\Phi'_{\ell_T})$ a.s.

Let $\omega\in\Omega_1$. Then from (1.2) we have that for $t_0, t\in[0,T]$

$$|\xi_t(\omega)[\phi] - \xi_{t_0}(\omega)[\phi]| \le |\gamma(\omega)[A_0T(0,t)\phi] - \gamma(\omega)[A_0T(0,t)\phi]|$$

$$+ |Y_t(\omega)[\phi] - Y_{t_0}(\omega)[\phi]| + |W_t(\omega)[\phi] - W_{t_0}(\omega)[\phi]|.$$

Hence from A1, A2, Lemma 1(b) and (1.7), for $m_T > \max(r_0, r, r_A, q)$

$$|\xi_t(\omega)[\phi] - \xi_{t_0}(\omega)[\phi]| \le \{2 \sup_{0\le t\le T} \|W_t(\omega)\|_{-q} + \|\gamma(\omega)\|_{-r_0}\} K_T \|\phi\|_{m_T}$$

for some constant K_T which does not depend on ω nor t and t_0.

Also from (1.7), (1.2) and the assumptions on W and η

$$E(\sup_{0\le t\le T} (\xi_t[\phi])^2) \le C_T^2 \|\phi\|^2_{m_T} \quad \text{for all } \phi\in\Phi$$

for some constant $C_T > 0$.

Let $\ell_T > m_T$ be such that the injection map $\Phi_{\ell_T} \hookrightarrow \Phi_{m_T}$ is Hilbert-Schmidt and let $\{\phi_j\}_{j\ge1} \subset \Phi$ be a CONS for Φ_{ℓ_T} with dual basis $\{\hat{\phi}_j\}_{j\ge1}$ a CONS for Φ'_{ℓ_T}. Then

$$E(\sum_{j=1}^{\infty} \sup_{0\le t\le T} (\xi_t[\phi_j])^2) < C_T^2 \sum_{j=1}^{\infty} \|\phi_j\|^2_{m_T} < \infty.$$

Let

$$\Omega_2 = \{\omega : \sum_{j=1}^{\infty} (\xi_t(\omega)[\phi_j])^2 < \infty\} \cap \Omega_1$$

and define $\tilde{\xi}_t(\omega) = \Sigma_{j=1}^{\infty}\xi_t(\omega)[\phi_j]\hat{\phi}_j$ for $\omega \in \Omega$, zero otherwise. Then

$$E(\sup_{0\le t\le T}\|\tilde{\xi}_t\|^2_{-\ell_T}) \le C_T^2 \sum_{j=1}^{\infty}\|\phi_j\|^2_{m_T} < \infty$$

and if $\omega \in \Omega_2$ and $t_0, t \in [0,T]$

$$\lim_{t\to t_0}\|\tilde{\xi}_t(\omega) - \tilde{\xi}_{t_0}(\omega)\|^2_{-\ell_T} = \lim_{t\to t_0}\sum_{j=1}^{\infty}(\xi_t(\omega)[\phi_j] - \xi_{t_0}(\omega)[\phi_j])^2$$

$$= \sum_{j=1}^{\infty}\lim_{t\to t_0}(\xi_t(\omega)[\phi_j] - \xi_{t_0}(\omega)[\phi_j])^2 = 0.$$

Then $\xi^T_\cdot(\omega) \in C([0,T],\Phi'_{\ell_T})$ $\omega \in \Omega_2$. Moreover

$$\tilde{\xi}_t(\omega)[\phi] = \sum_{j=1}^{\infty}\xi_t(\omega)[\phi_j]\hat{\phi}_j[\phi] = \sum_{j=1}^{\infty}\xi_t(\omega)[\phi_j]\langle\phi,\phi_j\rangle_{\ell_T}$$

$$= \sum_{j=1}^{\infty}\xi_t(\omega)[\langle\phi,\phi_j\rangle_{\ell_T}\phi_j] = \xi_t(\omega)[\phi] \quad \text{for all } \phi \in \Phi,\ 0 \le t \le T \quad \omega \in \Omega_2.$$

From now on we write ξ instead of $\tilde{\xi}$.

Hence we have shown that for each $T > 0$ there exists ℓ_T such that $\xi^T \in C([0,T];\Phi'_{\ell_T})$ a.s. i.e. $\xi^T \in C([0,T];\Phi')$ a.s. Then if $\Omega_T = \{\omega : \xi^T \in C([0,T],\Phi')\}$ $P(\Omega_T) = 1$ and taking $T_n \uparrow \infty$ and $\bar{\Omega} = \bigcap_{n=1}^{\infty}\Omega_n$ we have that for $\omega \in \bar{\Omega}$ $\xi(\omega) \in C(\mathbb{R}_+;\Phi')$, i.e. condition (6) in Definition 1 is satisfied.

Step 5. Uniqueness. Suppose that there exists a Φ'-valued process $\bar{\xi} = (\bar{\xi}_t)$ that is also a solution of (II). Then by Proposition 1 for each $T > 0$ there exists a set Ω_3 of probability one such that if $\omega \in \Omega_3$

$$\bar{\xi}^T_\cdot(\omega) \in C([0,T];\Phi'_{p_T}) \quad \text{some } p_T > \ell_T$$

and

$$\bar{\xi}_t(\omega)[\phi] = \gamma(\omega)[\phi] + \int_0^t \bar{\xi}_s(\omega)[A_s\phi]ds + W_t(\omega)[\phi] \quad \text{for all } \phi \in \Phi \quad 0 \le t \le T. \tag{1.8}$$

Fix $\omega \in \Omega_2 \cap \Omega_3$. Then, suppressing ω in the following, if in (1.8) we replace ϕ by $A_sT(s,t)\phi$ we have

$$W_s[A_sT(s,t)\phi] = \overline{\xi}_s[A_sT(s,t)\phi] - \gamma[A_sT(s,t)\phi] - \int_0^s \overline{\xi}_u[A_uA_sT(s,t)\phi]du.$$

Hence, substituting for $W_s[A_sT(s,t)\phi]$ in the expression on the RHS of (1.2) and using Fubini's theorem we have

$$\xi_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t \overline{\xi}_s[A_sT(s,t)\phi]ds - \int_0^t \gamma[A_sT(s,t)\phi]ds \tag{1.9}$$

$$- \int_0^t \int_u^t \overline{\xi}_u[A_uA_sT(s,t)\phi]dsdu + W_t[\phi].$$

Applying Lemma 2(b) to $F = \gamma$ and $B = I$ we have

$$\int_0^t \gamma[A_sT(s,t)\phi]ds = \gamma[T(0,t)\phi] - \gamma[\phi]. \tag{1.10}$$

Again applying Lemma 2(b) to $B = A_u$ and $F = \overline{\xi}_u$ we obtain

$$\int_u^t \overline{\xi}_u[A_uA_sT(s,t)\phi]ds = \overline{\xi}_u[A_uT(u,t)\phi] - \overline{\xi}_u[A_u\phi].$$

Finally using (1.10) and the above expression in (1.9) we have

$$\xi_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t \overline{\xi}_s[A_sT(s,t)\phi]ds - \gamma[T(0,t)\phi] + \gamma[\phi]$$

$$- \int_0^t \overline{\xi}_u[A_uT(u,t)\phi]du + \int_0^t \overline{\xi}_u[A_u\phi]du + W_t[\phi]$$

$$= \gamma[\phi] + \int_0^t \overline{\xi}_u[A_u\phi]du + W_t[\phi] = \overline{\xi}_t[\phi]$$

Thus for each $T > 0$

$$\xi_t(\omega)[\phi] = \overline{\xi}_t(\omega)[\phi] \quad \text{for all } \phi \in \Phi \quad 0 \le t \le T \quad \omega \in \Omega_2 \cap \Omega_3.$$

Then we have shown that for each $T > 0$ there exists a set Ω_T of probability one, such that for $\omega \in \Omega_T$ $\xi_t(\omega) = \overline{\xi}_t(\omega)$ $0 \le t \le T$. Let $T_n \uparrow \infty$ and define $\overline{\Omega} = \bigcap_{n=1}^{\infty} \Omega_{T_n}$, then

$$P(\xi_t = \overline{\xi}_t \ t \ge 0) = 1$$

which gives uniqueness of the solution. Q.E.D.

We now show the semimartingale and Gaussian property of the solution of the SDE(II).

A Φ'-valued stochastic process $(X_t)_{t\ge0}$ is said to be a Φ'-valued semimartinglae if for each $\phi \in \Phi$ $X_t[\phi]$ is a real valued semimartingale i.e.

$$X_t[\phi] = X_0^{\phi} + M_t^{\phi} + V_t^{\phi}$$

where M^{ϕ} is a real valued local martingale $M_0^{\phi} = 0$, V_t^{ϕ} a real valued right continuous adapted process whose paths are of finite variation, and X_0^{ϕ} is an $\mathcal{F}_0$-measurable random variable.

<u>Proposition 2</u>. Under the hypotheses of Theorem 1, the solution $\xi = (\xi_t)_{t\ge0}$ of the SDE(II) is a Φ'-valued semimartingale with canonical decomposition

$$\xi_t = W_t + \{T(0,t)'\gamma + \int_0^t T(s,t)'A_s'W_s ds\}.$$

<u>Proof</u>. From Theorem 1 we have that the solution of (II) is the Φ'-valued continuous stochastic process $\xi = (\xi_t)$ such that

$$(1.11) \qquad \xi_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t W_s[A_sT(s,t)\phi]ds + W_t[\phi] \quad \text{for all } \phi \in \Phi.$$

In step 3 of the proof of Theorem 1 it was shown that the Φ'-valued process

$$Y_t[\phi] = \int_0^t W_s[A_sT(s,t)\phi]ds$$

is continuous. Moreover it was also shown there, that if $0 \le t_0 \le t \le T$ then

$$(1.12) \qquad |Y_t[\phi] - Y_{t_0}[\phi]| \le \sup_{0\le s\le T} \|W_s\|_{-q} TD\|\phi\|_r (t - t_0).$$

Hence from (1.12) we have that for all $T > 0$ $Y_t(\omega)[\phi]$ is a process of finite variation for ω in a set of probability one.

Next define

$$g(t)[\phi] = \gamma[T(0,t)\phi].$$

Then using Kolmogorov's Forward equation

$$\frac{d}{dt}g(t)[\phi] = \frac{d}{dt}\gamma[T(0,t)\phi] = \gamma[\frac{d}{dt}T(0,t)\phi] = \gamma[T(0,t)A_t\phi].$$

Next defining $G_T(\phi) = \sup_{0\le t\le T}\|T(0,t)A_t\phi\|_{r_0}$ from Lemma 1.2(b) and assumptions A1 and A3(a), $G_T(\phi) < \infty$ for all $\phi \in \Phi$. Then using a Baire category argument

$$G_T(\phi) \le \theta_T\|\phi\|_{r_T} \quad \text{for all } \phi \in \Phi.$$

Hence, the function $[T(0,t)A_t\phi]$ is bounded in $(0,T)$ which implies that $g(t)$ is a function of bounded variation on $[0,T]$. Also clearly $g(t)$ is a continuous function of t.

Next define

$$V_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t W_s[A_sT(s,t)\phi]ds.$$

Then $V_t[\phi]$ is a Φ'-valued continuous process such that for all $\phi \in \Phi$ $V_t[\phi]$ has paths of bounded variation. Moreover since $\gamma[T(0,t)\phi]$ and $Y_t[\phi]$ are F_t-adapted then V_t is a predictable process. Hence we have the (unique) canonical decomposition

$$(1.13) \qquad \xi_t[\phi] = W_t[\phi] + V_t[\phi] \quad \text{for all } \phi \in \Phi \quad t \ge 0 \qquad \text{Q.E.D.}$$

<u>Proposition 3</u>. Assume the hypotheses of Theorem 1 and suppose that γ is a Φ'-valued Gaussian random variable independent of the Φ'-valued Wiener process $W = (W_t)_{t\ge0}$ with covariance Q. Then ξ_t is a Φ'-valued continuous Gaussian process with covariance

(1.14) $$K(\phi,\psi) = Q^{o}(T(0,t)\phi,T(0,t)\phi) + \int_0^t\int_0^t \min(s_1,s_2)Q(A_{s_1}T(s_1,t)\phi,A_{s_2}T(s_2,t)\psi)ds_1ds_2$$

$$+ Q(\phi,\psi) \qquad \phi,\psi \in \Phi \qquad t \geq 0$$

where Q^{o} is the covariance of γ.

Proof: Since $\int_0^t W_s[A_sT(s,t)\phi]ds$ and $W_t[\phi]$ are independent and $Y_t[\phi] = \int_0^t W_s[A_sT(s,t)\phi]ds$ is Gaussian with covariance

$$E(Y_t[\phi]Y_t[\psi]) = \int_0^t\int_0^t (\min(s_1,s_2))Q(A_{s_1}T(s_1,t)\phi,A_{s_2}T(s_2,t)\psi)ds_1ds_2$$

then the result follows since γ is independent of Y_t and W_t.

2. Solution of the SDE with Perturbation

In this section we solve the SDE (I).

Definition 2. We say that the SDE (I) has a Φ'-valued solution $\xi=(\xi_t)_{t\geq 0}$ if the following four conditions hold

a. (ξ_t) is F_t-adapted and Φ'-valued.

b. $\xi \in C(\mathbb{R}_+;\Phi')$ a.s.

c. $\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + \int_0^t \xi_s[P_s\phi]ds + W_t[\phi]$ for all $\phi \in \Phi$ a.s. $t \geq 0$.

d. For each $T > 0$

$$E(\sup_{0\leq t\leq T} |\xi_t[\phi]|^2) < \infty \quad \text{for all } \phi \in \Phi .$$

The following result is proved in the same way as Proposition 1.

Proposition 2. *If $\xi = (\xi_t)_{t\geq 0}$ is a solution of the SDE (II) then for each* $T > 0$ there exists $n_T > 0$ and a version of ξ (also denoted by ξ) such that

$$\xi^T_{\cdot} \in C([0,T];\Phi'_{n_T}) \ a.s.$$

and

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + \int_0^t \xi_s[P_s\phi]ds + W_t[\phi] \quad \text{for all } \phi \in \Phi, \quad 0 \leq t \leq T \text{ a.s.}$$

Remark. Condition (d) in Definition 2 is implied by the following one: For each $T > 0$

$$E\int_0^T (\xi_s[A_s\phi])^2 ds + E\int_0^T (\xi_s[P_s\phi])^2 ds < \infty.$$

In order to solve the SDE (I) we first solve the following stochastic equation:

(III) $$\xi_t = \int_0^t T'(s,t)P'_s\xi_s ds + \eta_t \quad t \geq 0$$

i.e.

$$\xi_t[\phi] = \int_0^t \xi_s[P_sT(s,t)\phi]ds + \eta_t[\phi] \quad \text{for all } \phi \in \Phi$$

where η_t is as in the following theorem. Then taking η_t as the solution given by Theorem 1 we obtain the solution of (I).

<u>Theorem 2</u>. Assume that A3(b)-(c) and A4 hold and let $\eta = (\eta_t)_{t\geq 0}$ be a Φ'-valued continuous stochastic process such that for each $T > 0$ there exists $q_T > 0$ and

$$E(\sup_{0\leq t\leq T} \|\eta_t\|^2_{-q_T}) < \infty.$$

Then there exists a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ of (III) on $C(\mathbb{R}_+; \Phi')$ with the following property: for each $T > 0$ there exists $p_T > 0$ such that

$$\xi^T_\cdot \in C([0,T];\Phi'_{p_T}) \text{ a.s., } E(\sup_{0\leq t\leq T} \|\xi_t\|^2_{-p_T} < \infty)$$

and

$$\xi_t[\phi] = \int_0^t \xi_s[P_s T(s,t)\phi]ds + \eta_t[\phi] \quad \text{for all } \phi \in \Phi \quad 0 \leq t \leq T \text{ a.s.}$$

<u>Proof</u>. (By successive approximations).

Let $T > 0$ fixed and

$$\Omega_4 = \{\omega : \sup_{0\leq t\leq T} \|\eta_t(\omega)\|_{-q_T} < \infty\}$$

Then $P(\Omega_4) = 1$.

Let $\omega \in \Omega_4$ and define for $0 \leq t \leq T$ the sequence of successive approximations:

$$\xi^0_t(\omega) = \eta_t(\omega)$$

$$\xi^1_t(\omega) = \int_0^t T(s,t)'P'_s\xi^0_s(\omega)ds + \eta_t(\omega)$$

$$\vdots$$

$$\xi^n_t(\omega) = \int_0^t T(s,t)'P'_s\xi^{n-1}_s(\omega)ds + \eta_t(\omega)$$

that is (suppressing ω in the writing) for $\phi \in \Phi$, $0 \leq t \leq T$ and $n \geq 1$

$$\xi_t^1[\phi] = \eta_t[\phi]$$

$$\xi_t^2[\phi] = \int_0^t \eta_s[P_s T(s,t)\phi]ds + \eta_t[\phi]$$

$$\vdots$$

$$\xi_t^n[\phi] = \int_0^t \xi_s^{n-1}[P_s T(s,t)\phi]ds + \eta_t[\phi].$$

Step 1. We shall prove that the above expressions are well defined elements of Φ' for all $n \geq 1$ and $t \geq 0$. Let

$$C_T = C_T(\omega) = \sup_{0\leq t\leq T} \|\xi_t^0(\omega)\|_{-q_T} = \sup_{0\leq t\leq T} \|\eta_t(\omega)\|_{-q_T} < \infty \,. \tag{2.1}$$

Using assumption A4, given $q_T > 0$ there exist positive constants $C_1 = C_1(T,q_T)$, $C_2 = C_2(T,q_T)$, $m_T = m_T(q_T)$ and $q_T' > q_T$ such that

$$\|\phi\|_{q_T} \leq C_1 |||\phi|||_{m_T} \leq C_2 \|\phi\|_{q_T'} \quad \text{for all} \quad \phi \in \Phi \,. \tag{2.2}$$

Also by assumptions A4(a)-(c) we have

$$\sup_{0\leq s\leq t\leq T} |||P_s T(s,t)\phi|||_{m_T} \leq K_T |||\phi|||_{m_T} \quad \text{for all } \phi \in \Phi. \tag{2.3}$$

Let $\omega \in \Omega_4$ and define (suppressing ω in the writing)

$$\xi_t^1[\phi] = \eta_t[\phi]$$

and for $n \geq 2$

$$\xi_t^2(\phi) = \int_0^t \xi_s^1[P_s T(s,t)\phi]ds + \eta_t[\phi] = \int_0^t \eta_s[P_s T(s,t)\phi]ds + \eta_t[\phi]$$

$$\xi_t^3(\phi) = \int_0^t \xi_{s_1}^2 (P_{s_1} T(s_1,t)\phi)ds + \eta_t[\phi]$$

$$= \int_0^t \int_0^{s_1} \eta_{s_2}[P_{s_2} T(s_2,s_1) P_{s_1} T(s_1,t)\phi]ds_2 ds_1 + \int_0^t \eta_{s_1} [P_{s_1} T(s_1,t)\phi]ds_1 + \eta_t[\phi]$$

$$\vdots$$

$$\xi_t^n(\phi) = \int_0^t \xi_s^{n-1}(P_{s_1} T(s_1,t)\phi)ds + \eta_t[\phi]$$

$$= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \eta_{s_{n-1}}[P_{s_{n-1}} T(s_{n-1},s_{n-2}) \cdots P_{s_1} T(s_1,t)\phi] ds_{n-1} \cdots ds_1 \tag{2.4}$$

$$+ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} \eta_{s_{n-2}}[P_{s_{n-2}} T(s_{n-2},s_{n-3}) \cdots P_{s_1} T(s_1,t)\phi] ds_{n-2} \cdots ds_1$$

$$+ \cdots + \eta_t[\phi].$$

Observe that the above integrals are well defined since using (2.2) and (2.3) we have

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \eta_{s_{n-1}}[P_{s_{n-1}} T(s_{n-1},s_{n-2}) \cdots P_{s_1} T(s_1,t)\phi] | ds_{n-1} \cdots ds_1$$

$$\le C_T \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \| P_{s_{n-1}} T(s_{n-1},s_{n-2}) \cdots P_{s_1} T(s_1,t)\phi \|_{q_T} ds_{n-1} \cdots ds_1$$

$$\le C_T C_1 \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} ||| P_{s_{n-1}} T(s_{n-1},s_{n-2}) \cdots P_{s_1} T(s_1,t)\phi |||_{m_T} ds_{n-1} \cdots ds_1 \tag{2.5}$$

$$\le C_T C_1 K_T \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} ||| P_{s_{n-2}} T(s_{n-2},s_{n-3}) \cdots P_{s_1} T(s_1,t)\phi |||_{m_T} ds_{n-2} \cdots ds_1$$

$$\le C_T C_1 \frac{(K_T)^n}{n!} t^n |||\phi|||_{m_T} \le C_T C_1 \frac{(K_T)^n}{n!} T^n |||\phi|||_{m_T} < \infty$$

Then each integral in (2.4) is well defined and furthermore from the second inequality in (2.2), for all $n \ge 1$ and $0 \le t \le T$ we have

$$|\xi_t^n(\omega)(\phi)| \le C_T(\omega) C_1 \left(\sum_{k=0}^n \frac{(K_T T)^k}{k!} \right) C_2 \|\phi\|_{q_T'} \quad \text{for all } \phi \in \Phi \tag{2.6}$$

Then for each $n \ge 1$ and $0 \le t \le T$ $\xi_t^n(\omega) \in \Phi_{q_T'}$,

$$\|\xi_t^n(\omega)\|_{-q_T'} \le C_T(\omega) C_1 C_2 \sum_{k=0}^n \frac{(K_T T)^k}{k!} \le C_T C_1 C_2 e^{K_T T} \tag{2.7}$$

and we can write $\xi_t^n(\omega)[\phi] = \xi_t^n(\omega)(\phi) \quad \omega \in \Omega_3$.

Step 2. The sequence (ξ_t^n) converges.

From (2.4) we have that if $m \le n$

$$\xi_t^n[\phi] - \xi_t^m[\phi] = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \eta_{s_{n-1}} [P_{s_{n-1}} T(s_{n-1}, s_{n-2}) \cdots P_{s_1} T(s_1, t)\phi] ds_n \cdots ds_1$$

$$+ \cdots + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-1}} \eta_{s_m} [P_{s_m} T(s_m, s_{m-1}) \cdots P_{s_1} T(s_1, t)\phi] ds_m \cdots ds_1$$

and proceeding as in (2.5)

$$|\xi_t^n[\phi] - \xi_t^m[\phi]| \le C_T C_1 C_2 \sum_{k=m+1}^{n} \frac{(K_T T)^n}{k!} \|\phi\|_{q_T'} \quad \text{for all } \phi \in \Phi. \tag{2.8}$$

Then for each $\phi \in \Phi$ and $\omega \in \Omega_3$ $|\xi_t^n[\phi] - \xi_t^m[\phi]|$ converges to zero uniformly on $[0,T]$ as $n > m \to \infty$. Hence $\{\xi_t^n(\omega)[\phi]\}_{n \ge 1}$ is a Cauchy sequence of real numbers and from (2.6) for $0 \le t \le T$

$$|\lim \xi_t^n(\omega)[\phi]| \le C_T(\omega) C_1 C_2 e^{K_T T} \|\phi\|_{q_T'} \quad \text{for all } \phi \in \Phi.$$

Hence for $0 \le t \le T$ and $\omega \in \Omega_3$ $\xi_t(\omega)$ defined by

$$\xi_t(\omega)[\phi] = \lim_{n \to \infty} \xi_t^n(\omega)[\phi] \quad \text{for all } \phi \in \Phi$$

is such that

$$\sup_{0 \le t \le T} |\xi_t(\omega)[\phi]| \le C_T(\omega) C_1 C_2 e^{K_T T} \|\phi\|_{q_T'} \quad \text{for all } \phi \in \Phi$$

and therefore $\xi_t(\omega)$ is a linear functional on Φ, i.e. $\xi_t(\omega) \in \Phi'$. Moreover from the last expression we have that ξ satisfies (d) in Definition 2.

Next let $\ell_T > q_T'$ be such that the injection map $\Phi_{\ell_T} \hookrightarrow \Phi_{q_T'}$ is Hilbert-Schmidt and let $\{\phi_j\}_{j \ge 1} \in \Phi$ be a CONS for Φ_{ℓ_T} with dual basis $\{\hat{\phi}_j\}_{j \ge 1}$ CONS in Φ'_{ℓ_T}. Then

$$\sup_{0\le t\le T} \sum_{j=1}^{\infty} |\xi_t(\omega)[\phi_j]|^2 \le C_T^2(\omega) C_1^2 C_2^2 e^{2K_T T} \sum_{j=1}^{\infty} \|\phi_j\|_{q_T'}^2 < \infty$$

and we can define

$$\tilde{\xi}_t(\omega) = \sum_{j=1}^{\infty} \xi_t(\omega)[\phi_j]\hat{\phi}_j .$$

Hence $\tilde{\xi}_t(\omega) \in \Phi'_{\ell_T}$ $0 \le t \le T$ $\omega \in \Omega_3$ and moreover $\tilde{\xi}_t(\omega)[\phi] = \xi_t(\omega)[\phi]$ for all $\phi \in \Phi$:

$$\begin{aligned}
\tilde{\xi}_t(\omega)[\phi] &= \sum_{j=1}^{\infty} \xi_t(\omega)[\phi_j]\hat{\phi}_j[\phi] = \sum_{j=1}^{\infty} \xi_t(\omega)[\phi_j]\langle\phi,\phi_j\rangle_{\ell_T} \\
&= \sum_{j=1}^{\infty} \xi_t(\omega)[\langle\phi,\phi_j\rangle_{\ell_T}\phi_j] = \lim_{n\to\infty}\xi_t(\omega)\Big[\sum_{j=1}^{n}\langle\phi,\phi_j\rangle_{\ell_T}\phi_j\Big] \\
&= \xi_t(\omega)[\phi].
\end{aligned}$$

<u>Step 3</u>. ξ_t satisfies (c) in Definition 2, i.e.

$$P\Big(\omega : \xi_t(\omega)[\phi] = \int_0^t \xi_s(\omega)[P_s T(s,t)\phi]ds + \eta_t(\omega)[\phi] \quad \text{for all } \phi\Big) = 1, \quad 0 \le t \le T.$$

Let $\omega \in \Omega_3$, $\phi \in \Phi$ and $0 \le t \le T$. Then

$$\xi_t^n(\omega)[\phi] = \int_0^t \xi_s^{n-1}(\omega)[P_s T(s,t)\phi]ds + \eta_t[\phi].$$

Next, by assumptions (A4)(a)-(b) and Remark 1, given $\ell_T > 0$ there exist positive constants d_1, d_2, m' and ℓ_T' such that

$$(2.9) \qquad \|\phi\|_{\ell_T} \le d_1 |||\phi|||_{m'} \le d_2 \|\phi\|_{\ell_T'} \quad \text{for all } \phi \in \Phi .$$

Then

$$\begin{aligned}
\sup_{0\le s\le t\le T} \|P_s T(s,t)\phi\|_{\ell_T} &\le d_1 \sup_{0\le s\le t\le T} |||P_s T(s,t)\phi|||_{m'} \\
&\le d_1 D |||\phi|||_{m'}
\end{aligned}$$

and using (2.7) since $\ell_T > q'_T$

$$|\xi_s^{n-1}[P_s T(s,t)\phi]| \le \|\xi_s^{n-1}\|_{-\ell_T} \|P_s T(s,t)\phi\|_{\ell_T}$$

$$\le C_T C_1 C_2 e^{K_T T} d_1 D |||\phi|||_m, < \infty \text{ for all } n \ge 1,\ 0 \le s \le t \le T.$$

Then using dominated convergence theorem

$$\xi_t[\phi] = \lim_{n\to\infty} \xi_t^n[\phi] = \lim_{n\to\infty} \int_0^t \xi_s^{n-1}[P_s T(s,t)\phi]ds + \eta_t[\phi]$$

$$= \int_0^t \xi_s[P_s T(s,t)\phi]ds + \eta_t[\phi] \quad \text{for all } \phi \in \Phi,\ 0 \le t \le T.$$

Step 4. $\xi_\cdot^T \in C([0,T];\Phi'_{p_T})$ some $p_T > 0$. Let $t_0, t \in [0,T]$ $T > 0$. Assume $t_0 < t$, then

$$\xi_t[\phi] - \xi_{t_0}[\phi] = \int_0^t \xi_s[P_s T(s,t)\phi]ds - \int_0^{t_0} \xi_s[P_s T(s,t_0)\phi]ds + \eta_t[\phi] - \eta_{t_0}[\phi]. \tag{2.10}$$

But

$$\int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu - \int_0^{t_0} \xi_\mu[P_\mu T(\mu,t_0)]d\mu$$

$$= \int_0^{t_0} (\xi_\mu[P_\mu T(\mu,t)\phi] - \xi_\mu[P_\mu T(\mu,t_0)\phi])d\mu + \int_{t_0}^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu.$$

Next using Lemma 2(a) with $F = \xi_\mu$, $B = P_\mu$, we obtain

$$\xi_\mu[P_\mu T(\mu,t)\phi] = \xi_\mu[P_\mu\phi] + \int_\mu^t \xi_\mu[P_\mu T(\mu,s)A_s\phi]ds$$

and again applying Lemma 2(a) to $F = \xi_\mu$, $B = P_\mu$, $t = t_0$

$$\xi_\mu[P_\mu T(\mu,t_0)\phi] = \xi_\mu[P_\mu\phi] + \int_\mu^{t_0} \xi_\mu[P_\mu T(\mu,s)A_s\phi]ds$$

and therefore

$$\xi_\mu[P_\mu T(\mu,t)\phi] - \xi_\mu[P_\mu T(\mu,t_0)] = \int_{t_0}^t \xi_\mu[P_\mu T(\mu,s)A_s\phi]ds.$$

Hence,

$$
\int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu - \int_0^{t_0} \xi_\mu[P_\mu T(\mu,t_0)\phi]d\mu
$$

(2.11)

$$
= \int_0^{t_0} \int_{t_0}^{t} \xi_\mu[P_\mu T(\mu,s)A_s\phi]dsd\mu + \int_{t_0}^{t} \xi_\mu[P_\mu T(\mu,t)\phi]d\mu.
$$

By assumption A4(c) for $m_1 \geq n_T$ some $n_T > 0$ and $T > 0$

$$
\sup_{0\leq s\leq t\leq T} |||P_s T(s,t)\phi|||_{m_1} \leq K_1(m_1,T)|||\phi|||_{m_1} \quad \text{for all } \phi \in \Phi
$$

and using assumption A3(a) and a Baire category argument, for some $m' \geq m_1$ and $K(m',T) > 0$ we have

$$
\sup_{0\leq s\leq t\leq T} |||P_s T(s,t)A_s\phi|||_{m'} \leq K(m',T)|||\phi|||_{m'} \quad \text{for all } \phi \in \Phi.
$$

Moreover $\sup_{0\leq s\leq t}||\xi_s||_{-\ell_T} < \infty$ since from (2.7)

$$
\sup_{0\leq t\leq T} ||\xi_t^n||_{-\ell_T} \leq \sup_{0\leq t\leq T} ||\xi_t^n||_{-q_T'} \leq C_T C_1 C_2 e^{TK_T} < \infty.
$$

Hence, using the last three expressions and (2.9) in (2.11)

$$
|\int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu - \int_0^{t_0} \xi_\mu[P_\mu T(\mu,t_0)\phi]d\mu
$$

(2.12)

$$
\leq \int_0^{t_0}\int_{t_0}^{t} ||\xi_\mu||_{-\ell_T} ||P_\mu T(\mu,s)A_s\phi||_{\ell_T} d\mu ds + \int_{t_0}^{t} ||\xi_\mu||_{-\ell_T} ||P_\mu T(\mu,t)\phi||d\mu
$$

$$
\leq C_T D_T ||\phi||_{\ell_T'}(t - t_0) \qquad 0 \leq t_0 \leq t \leq T
$$

where $D_T = d_2 C_1 C_2 T e^{TK_T}$. Then for all $t, t_0 \in [0,T]$

$$
|\int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu - \int_0^{t_0} \xi_\mu[P_\mu T(\mu,t_0)\phi]d\mu| \leq |t - t_0| C_T D_T ||\phi||_{\ell_T'}.
$$

Then from the last expression $Z_t[\phi] = \int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu$ is a continuous process in $t \in [0,T]$ for each $\phi \in \Phi$. Then from (2.10) we obtain that $\xi_t[\phi]$ is also a continuous process in $t \in [0,T]$ for each $\phi \in \Phi$. Moreover

$$(2.13) \qquad \sup_{0\le t\le T} |\xi_t[\phi]| \le (C_T D_T + C_T)\|\phi\|_{\ell'_T}.$$

Next let $p_T > \ell'_T$ be such that the injection map $\Phi_{p_T} \hookrightarrow \Phi_{\ell'_T}$ is Hilbert-Schmidt and let $\{e_j\}_{j\ge 1} \subset \Phi$ be a CONS for Φ_{p_T} with dual basis $\{\hat{e}_j\}_{j\ge 1}$ a CONS for Φ'_{p_T}. Then from (2.13) we have

$$(2.14) \qquad \sup_{0\le t\le T} \sum_{j=1}^{\infty} |\xi_t^{(\omega)}[e_j|^2 \le (C_T D_T + C_T)^2 \sum_{j=1}^{\infty} \|e_j\|^2_{\ell'_T} < \infty.$$

Hence, define $\tilde{\xi}_t(\omega) = \Sigma_{j=1}^{\infty} \xi_t[e_j]\hat{e}_j$ which is an element in Φ'_{p_T} and $\tilde{\xi}_t(\omega)[\phi] = \xi_t(\omega)[\phi]$ for all $\phi \in \Phi$ $0 \le t \le T$ $\omega \in \Omega_3$. Then by dominated convergence theorem, since $\xi_t(\omega)[e_j]$ is continuous in t for each $j \ge 1$ we have that

$$\lim_{t\to t_0} \|\tilde{\xi}_t - \tilde{\xi}_{t_0}\|_{-p_T} = \lim_{t\to t_0} \sum_{j=1}^{\infty} (\xi_t[e_j] - \xi_{t_0}[e_j])^2$$

$$= \sum_{j=1}^{\infty} \lim_{t\to t_0} (\xi_t[e_j] - \xi_{t_0}[e_j])^2 = 0 \qquad t_0 \in [0,T].$$

Then $\xi^T_\cdot(\omega) \in C([0,T];\Phi'_{p_T})$ for some $p_T > 0$ $\omega \in \Omega_3$, $P(\Omega_3) = 1$. Moreover from (2.13), (2.1) and the assumption on η_t we have

$$E(\sup_{0\le t\le T} |\xi_t[\phi]|^2) < \infty .$$

Furthermore from (2.14) and since by assumption on η_t $E(C_T^2) < \infty$ we have that

$$E(\sup_{0\le t\le T} \|\xi\|^2_{-p_T}) \le E(C_T D_T + C_T)^2 \sum_{j=1}^{\infty} \|\phi\|^2_{-\ell_T} < \infty.$$

A similar argument to that at the end of Step 4 in Theorem 1 gives that $\xi \in C(\mathbb{R}_+,\Phi')$ a.s.

Step 5 Uniqueness

To show uniqueness let X be any solution of (5.3). For the present assume that X_t satisfies the following condition:

(*) For each $T > 0$ there exists $p_T' > 0$ such that $X_\cdot^T \in C([0,T];\Phi'_{p_T'})$ a.s.

WLOG let $p_T' > p_T$ and

$$\Omega_4 = \{\omega : \sup_{0 \le t \le T} \|X_t\|_{-p_T'} < \infty\}.$$

Then $P(\Omega_4) = 1$. Fix $\omega \in \Omega_3 \cap \Omega_4$ and let $0 \le t \le T$. Then for each $\phi \in \Phi$ (suppressing ω in the writing)

$$X_t[\phi] = \int_0^t X_s[P_s T(s,t)\phi]ds + \eta_t[\phi].$$

Next if ξ_t^n is the sequence of successive approximations defined in (2.4) we have that for $0 \le t \le T$ and $\phi \in \Phi$

$$X_t[\phi] - \xi_t^1[\phi] = \int_0^t X_s[P_s T(s,t)\phi]ds \tag{2.15}$$

$$X_t[\phi] = \xi_t^2[\phi] = \int_0^t X_s[P_s T(s,t)\phi]ds - \int_0^t \xi_s^1[P_s T(s,t)\phi]ds$$

$$\vdots$$

$$X_t[\phi] - \xi_t^n[\phi] = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \{X_{s_n}[P_{s_n} T(s_n, s_{n-1}) \cdots P_{s_1} T(s_1,t)\phi]$$

$$- \xi_{s_n}^1 [P_{s_n} T(s_n, s_{n-1}) \cdots P_{s_1} T(s_1,t)\phi]\} ds_n \cdots ds_1$$

$$= \int_0^t \int_0^{s_1} \cdots \int_0^{s_n} X_{s_{n+1}} [P_{s_{n+1}} T(s_{n+1}, s_n) \cdots P_{s_1} T(s_1,t)\phi] ds_{n+1} \cdots ds_1 .$$

Then using the inequalities similar to (2.2) and (2.3) it follows that

$$X_t[\phi] - \xi_t^n[\phi] \le \int_0^t \cdots \int_0^{s_n} \|X_{s_{n+1}}\|_{-p_T} \|P_{s_{n+1}} T(s_{n+1}, s_n) \cdots P_{s_1} T(s_1,t)\|_{p_T} ds_n \cdots ds_1$$

$$\leq \sup_{0\leq t\leq T} \|X_t\|_{-p_T} C_1' C_2' \frac{(K_T' T)^n}{n!} \|\phi\|_{m_T'} < \infty \quad \text{for all } \phi \in \Phi$$

for some positive constants C_1', C_2', K_T' and m_T'.

Hence

$$\sup_{0\leq t\leq T} |X_t[\phi] - \xi_t^n[\phi]| \to 0 \quad \text{as } n \to \infty.$$

Thus $P(X_t = \xi_t \quad 0 \leq t \leq T) = 1$ and a similar argument to that at the end of Step 5 in Theorem 1 gives $P(X_t = \xi_t \quad t \geq 0) = 1$. The proof of the theorem is complete.

Q.E.D.

Using Theorems 1 and 2 we now solve the SDE(I) i.e.

$$\text{(I)} \qquad \begin{cases} d\xi_t = (A_t' + P_t')\xi_t dt + dW_t \\ \xi_0 = \gamma \end{cases}$$

<u>Theorem 3</u>. Under assumptions A1-A4 the SDE (I) has a unique solution $\xi = (\xi_t)_{t\geq 0}$ such that for each $T > 0$ there exists $p_T > 0$ and

$$\xi_\cdot^T \in C([0,T];\Phi_{p_T}') \quad \text{a.s.}$$

and

$$E(\sup_{0\leq t\leq T} \|\xi_t\|_{-p_T}^2) < \infty.$$

<u>Proof</u>. Let η_t be the solution of the SDE

$$d\eta_t = A_t'\eta_t dt + dW_t$$

$$\eta_0 = \gamma$$

whose unique solution is given by Theorem 1 and it is such that for each $T > 0$ there exists $\ell = \ell_T > \max(q, r_0)$,

$$\eta_\cdot^T \in C([0,T];\Phi_{\ell_T}') \quad \text{a.s.}$$

and

$$(2.16) \qquad \eta_t[\phi] = \int_0^t \eta_s[A_s\phi]ds + W_t[\phi] + \gamma[\phi] \quad \text{for all } \phi \in \Phi \quad 0 \le t \le T \quad \text{a.s.}$$

Let $\xi = (\xi_t)$ be the solution, given by Theorem 2, of the SDE

$$(2.17) \qquad \xi_t = \int_0^t T'(s,t)P_s'\xi_s ds + \eta_t$$

which is such that for each $T > 0$ there exists $m_T > \ell_T$ such that

$$\xi_\cdot^T \in C([0,T];\Phi'_{m_T}) \quad \text{a.s.}$$

and

$$(2.18) \qquad \xi_t[\phi] = \int_0^t \xi_s[P_sT(s,t)\phi]ds + \eta_t[\phi] \quad \text{for all } \phi \in \Phi \quad 0 \le t \le T \quad \text{a.s.}$$

We shall prove that ξ_t is the unique solution of (I). First we show that it is a solution of (I):

Applying Lemma 2(a) to $B = P_\mu$ and $F = \xi_\mu$ we have

$$(2.19) \qquad \xi_\mu[P_\mu T(\mu,t)\phi] = \xi_\mu[P_\mu\phi] + \int_\mu^t \xi_\mu[P_\mu T(\mu,s)A_s\phi]ds.$$

Let

$$\Omega_1 = \{\omega : \xi_\cdot^T \in C([0,T];\Phi'_{\ell_T}\}$$

$$\Omega_2 = \{\omega : \xi_\cdot^T \in C([0,T];\Phi'_{m_T}\}$$

then $P(\Omega_1) = P(\Omega_2) = 1$. Let $\omega \in \Omega_1 \cap \Omega_2$ then (suppressing ω in the writing) integrating (2.19) and applying Fubini's theorem we have

$$(2.20) \qquad \int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu = \int_0^t \xi_\mu[P_\mu\phi]d\mu + \int_0^t \int_\mu^t \xi_\mu[P_\mu T(\mu,s)A_s\phi]dsd\mu$$

$$= \int_0^t \xi_\mu[P_\mu\phi]d\mu + \int_0^t \int_0^s \xi_\mu[P_\mu T(\mu,s)A_s\phi]d\mu ds.$$

Next from (2.18)

$$\xi_s[A_s\phi] = \int_0^s \xi_\mu[P_\mu T(\mu,s)A_s\phi]d\mu + \eta_s[A_s\phi].$$

Then using the above expression in the second term of (2.20) we obtain

$$(2.21)\qquad \int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu = \int_0^t \xi_\mu[P_\mu\phi]d\mu + \int_0^t \xi_s[A_s\phi]ds - \int_0^t \eta_s[A_s\phi]ds.$$

But also from (2.18)

$$\int_0^t \xi_\mu[P_\mu T(\mu,t)\phi]d\mu = \xi_t[\phi] - \eta_t[\phi].$$

Hence from the above expression and (2.21) we obtain that

$$\xi_t[\phi] - \eta_t[\phi] = \int_0^t \xi_s[P_s\phi]ds + \int_0^t \xi_s[A_s\phi]ds - \int_0^t \eta_s[A_s\phi]ds$$

i.e.

$$(2.22)\qquad \xi_t[\phi] = \int_0^t \xi_s[P_s\phi]ds + \int_0^t \xi_s[A_s\phi]ds + \eta_t[\phi] - \int_0^t \eta_s[A_s\phi]ds.$$

But $\eta_t[\phi] - \int_0^t \eta_s[A_s\phi]ds = \gamma[\phi] + W_t[\phi]$, then

$$\xi_t[\phi] = \int_0^t \xi_s[P_s\phi]ds + \int_0^t \xi_s[A_s\phi]ds + \gamma[\phi] + W_t[\phi] \quad \text{for all } \phi \in \Phi$$

i.e.

$$d\xi_t = (A_t' + P_t')\xi_t dt + dW_t.$$

Now we shall show that the solution $\xi = (\xi_t)$ of (I) is unique. Suppose there exists a Φ'-valued process $\bar{\xi}_t$ that is also a solution of (I). Then by Proposition 4 for each $T > 0$ there exists a set Ω_3 of probability one such that if $\omega \in \Omega_3$

$$\xi^T_\cdot(\omega) \in C([0,T];\Phi'_{q_T}) \quad \text{some } q_T > 0$$

and

$$(2.23) \qquad \bar{\xi}_t(\omega)[\phi] = \int_0^t \bar{\xi}_s(\omega)[P_s\phi]ds + \int_0^t \bar{\xi}_s(\omega)[A_s\phi]ds + \gamma(\omega)[\phi] + W_t[\phi]$$

$$\text{for all } \phi \in \Phi \quad 0 \le t \le T.$$

Fix $\omega \in \Omega_4 = \Omega_1 \cap \Omega_2 \cap \Omega_3$. Then (suppressing ω in the writing) we have that for $0 \le s \le t \le T$ and $\phi \in \Phi$

$$(2.24) \qquad W_s[A_sT(s,t)\phi] = \bar{\xi}_s[A_sT(s,t)\phi] - \int_0^s \bar{\xi}_\mu[P_\mu A_sT(s,t)\phi]d\mu$$

$$- \int_0^t \bar{\xi}_\mu[A_\mu A_sT(s,t)\phi]d\mu - \gamma[A_sT(s,t)\phi].$$

On the other hand from (2.17), (2.16) and Theorem 1 we have that for $0 \le t \le T$ and $\phi \in \Phi$

$$(2.25) \qquad \xi_t[\phi] - \int_0^t \xi_s[P_sT(s,t)\phi]ds = \int_0^t W_s[A_sT(s,t)\phi]ds + \gamma[T(0,t)\phi] + W_t[\phi].$$

Hence, using (2.24) in (2.25) we have that

$$(2.26) \qquad \xi_t[\phi] - \int_0^t \xi_s[P_sT(s,t)\phi]ds = \int_0^t \bar{\xi}_s [A_sT(s,t)\phi]ds$$

$$- \int_0^t\int_0^s \bar{\xi}_\mu [P_\mu A_sT(s,t)\phi]d\mu ds - \int_0^t\int_0^s \bar{\xi}_\mu [A_\mu A_sT(s,t)\phi]d\mu ds$$

$$- \int_0^t \gamma[A_sT(s,t)\phi]ds + \gamma[T(0,t)\phi] + W_t[\phi].$$

Next, using Lemma 2(b) with $F = \gamma$ and $B = I$ we obtain

$$(2.27) \qquad - \int_0^t \gamma[A_sT(s,t)\phi]ds + \gamma[T(0,t)\phi] = \gamma[\phi].$$

Again, applying Lemma 2(b) to $F = \bar{\xi}_\mu$, $B = P_\mu$ and to $F = \bar{\xi}_\mu$ and $B = A_\mu$ we obtain the following two expressions

$$(2.28) \qquad -\int_0^t \bar{\xi}_\mu[P_\mu A_sT(s,t)\phi]ds = \bar{\xi}_\mu[P_\mu\phi] - \bar{\xi}_\mu[P_\mu T(\mu,t)\phi]$$

and

$$(2.29)\qquad -\int_0^t \bar{\xi}_\mu[A_\mu A_s T(s,t)\phi]ds = \bar{\xi}_\mu[A_\mu\phi] - \bar{\xi}_\mu[A_\mu T(\mu,t)\phi].$$

Hence, using (2.27), (2.28) and (2.29) in (2.26), we obtain

$$\xi_t[\phi] - \int_0^t \xi_s[P_s T(s,t)\phi]ds = \int_0^t \bar{\xi}_s[A_s T(s,t)\phi]ds + \int_0^t \bar{\xi}_\mu[P_\mu\phi]d\mu$$

$$- \int_0^t \bar{\xi}_\mu[P_\mu T(\mu,t)\phi]d\mu + \int_0^t \bar{\xi}_\mu[A_\mu\phi]d\mu - \int_0^t \bar{\xi}_\mu[A_\mu T(\mu,t)\phi]d\mu + \gamma[\phi] + W_t[\phi],$$

that is

$$\xi_t[\phi] - \int_0^t \xi_s[P_s T(s,t)\phi]ds = \int_0^t \bar{\xi}_\mu[P_\mu\phi]d\mu + \int_0^t \bar{\xi}_\mu[A_\mu\phi]d\mu + \gamma[\phi] + W_t[\phi]$$

$$- \int_0^t \bar{\xi}_\mu[P_\mu T(\mu,t)\phi]d\mu.$$

Hence using (2.23) for $\omega \in \Omega_4$ $0 \le t \le T$ and $\phi \in \Phi$ we have that

$$\bar{\xi}_t[\phi] - \int_0^t \bar{\xi}_s[P_s T(s,t)\phi]ds = \eta_t[\phi].$$

Thus any solution $\bar{\xi}_t$ of (I) is a solution of (II) and therefore since Proposition 4 implies condition (*) in Step 5 of Theorem 2, the solution of (I) is unique.

Then the properties of $\xi = (\xi_t)$ follow from Theorems 1 and 2 and the proof of Theorem 3 is complete.

Q.E.D.

<u>Proposition 5</u>. Under the hypotheses of Theorem 3, the solution $\xi = (\xi_t)$ of the SDE (III) is a Φ'-valued continuous semimartingale with canonical decomposition

$$\xi_t = W_t + \{T'(0,t)\gamma + \int_0^t T'(s,t)A'_s W_s ds + \int_0^t T'(s,t)P'_s\xi_s ds\}.$$

<u>Proof</u>. From the proof of Theorem 3 $\xi = (\xi_t)$ is such that for all $t \ge 0$ and $\phi \in \Phi$

$$\xi_t[\phi] = \int_0^t \xi_s[P_s T(s,t)\phi] + \eta_t[\phi] + \gamma[T(0,t)\phi]$$

where

$$\eta_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t W_s[A_s T(s,t)\phi]ds + W_t[\phi]$$

and from Proposition 2 η_t is a Φ'-valued semimartingale with canonical decomposition

$$\eta_t = W_t + V_t^1$$

$$V_t^1 = T'(0,t)\gamma + \int_0^t T'(s,t)A_s'W_s ds.$$

Hence it only remains to prove that

$$Z_t[\phi] = \int_0^t \xi_s[P_s T(s,t)\phi]ds$$

is a process of bounded variation. It was shown in Step 4 of the proof of Theorem 2 that $Z_t[\phi]$ is continuous in t for each $\phi \in \Phi$ on a set of probability one. Moreover from (2.12) we have that for each $T > 0$ and $0 \le t_0 \le t \le T$

$$|Z_t[\phi] - Z_{t_0}[\phi]| \le C_T D_T T \|\phi\|_{\ell_T'}(t - t_0).$$

Hence the process $Z_t[\phi]$ is of bounded variation on $[0,T]$ for each $T > 0$. Moreover since it is continuous and F_t-adapted, it is predictable.

Writing

$$V_t[\phi] = Z_t[\phi] + V_t^1[\phi]$$

we have that $V_t[\phi]$ is a continuous predictable process of finite variation and $\xi_t[\phi]$ admits the canonical decomposition

$$\xi_t[\phi] = W_t[\phi] + V_t[\phi].$$

Q.E.D.

Proposition 6. Under the hypothesis of Proposition 5 if γ is as in Proposition 3 then the solution $\xi = (\xi_t)_{t \ge 0}$ of the SDE (III) is a Φ'-valued continuous Gaussian process.

Proof. From the proof of Theorem 2 $\xi_t[\phi]$ is the a.s. limit of a sequence of Gaussian random variables $\xi_t^n[\phi]$. Hence $\xi_t[\phi]$ is Gaussian.

Q.E.D.

3. Stochastic Evolution Equations Driven By Nuclear Space Valued Martingales

A Φ'-valued stochastic process $M = (M_t)_{t\geq 0}$ is a Φ'-valued martingale with respect to a right continuous filtration $(F_t)_{t\geq 0}$ if for each $\phi \in \Phi$ $M_t[\phi]$ is a real valued martingale with respect to (F_t). In this section the following result will be useful.

Proposition 7. If M is a Φ'-valued martingale with respect to F_t then there exists a Φ'-valued version $\tilde{M}$ of M such that the following two conditions hold:

a. For each $T > 0$ there exists $q_T > 0$ such that

$$\tilde{M}^T_{\cdot} \in D([0,T];\Phi'_{q_T}) \text{ a.s.},$$

where $D([0,T];\Phi'_{q_T})$ is the Skorohod space of right continuous left hand limits (r.c.l.l.) functions from $[0,T]$ to Φ'_{q_T}.

b. $\tilde{M}$ is r.c.l.l. in the strong Φ'-topology, i.e.

$$\tilde{M} \in D([0,\infty);\Phi') \text{ a.s.}$$

For the proof of this proposition, see Mitoma (1981).

Consider the stochastic evolution equation

$$\text{(IV)} \qquad \begin{cases} d\xi_t = A'_t\xi_t dt + P'_t\xi_t dt + dM_t \\ \xi_0 = \gamma \end{cases}$$

where γ, A_t and P_t are as in assumptions A1,A3 and A4 in the Introduction and M_t is a Φ'-valued right continuous martingale such that $E(M_t[\phi])^2 < \infty$ for all $\phi \in \Phi$, $t \geq 0$.

In this section we show how to solve the SDE (IV) in a similar manner as for the Φ'-valued Wiener case. Our goal is to prove the following analog of Theorem 3.

<u>Theorem 6</u>. Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued martingale such that $E(M_t[\phi])^2 < \infty$ for all $\phi \in \Phi$ and assume that A1, A3 and A4 hold. Then the SDE (IV) has a unique solution $\xi = (\xi_t)_{t\geq 0}$ such that for each $T > 0$ there exists $p_T > 0$ and

$$\xi_{\cdot}^T \in D([0,T];\Phi'_{p_T})$$

and

$$E(\sup_{0\leq t\leq T} ||\xi_t||^2_{-p_T}) < \infty.$$

Furthermore ξ is a Φ'-valued semimartingale with decomposition

$$\xi_t = \{T'(0,t)\gamma + \int_0^t T'(s,t)A'_s M_s ds + \int_0^t T'(s,t)P'_s \xi'_s ds\} + M_t.$$

As in the Φ'-valued Wiener case we first solve the SDE without perturbation.

<u>Theorem 7</u>. Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued martingale such that $E(M_t[\phi])^2 < \infty$ for all $\phi \in \Phi$ and assume that A1 and A3 hold. Then the SDE

$$\text{(V)} \qquad \begin{cases} d\xi_t = A'_t \xi'_t dt + dM_t \\ \xi_0 = \gamma \end{cases}$$

has a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ given by

1. $\xi_t = T'(0,t)\gamma + \int_0^t T'(s,t)A'_s M_s ds + M_t \quad t \geq 0$

i.e.

$$\text{(3.1)} \qquad \xi_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t M_s[A_s T(s,t)\phi]ds + M_t[\phi] \quad \text{for all } \phi \in \Phi \quad t \geq 0 \text{ a.s.}$$

Furthermore ξ_t satisfies the following two conditions:

2. for each $T > 0$ there exists $\ell_T > 0$ such that

$$\xi_{\cdot}^T \in D([0,T];\Phi'_{\ell_T}) \quad \text{a.s.}$$

and

$$E(\sup_{0\leq t\leq T} ||\xi_t||^2_{-\ell_T}) < \infty .$$

3. ξ_t is a Φ'-valued semimartingale with decomposition

$$\xi_t = \{T'(0,t)\gamma + \int_0^t T'(s,t)A_s'M_s ds\} + M_t .$$

Proof. Since the proof of this theorem is very similar to that of Theorem 1 we only give an outline of it.

Let $T > 0$, then by Proposition 7(a) there exists $q_T > 0$ such that

$$M_\cdot^T \in D([0,T];\Phi'_{q_T}) \quad \text{a.s.}$$

Let

$$\Omega_1^T = \{\omega : M_\cdot^T(\omega) \in D([0,T];\Phi'_{q_T})\} \cap \{\omega : \|\gamma(\omega)\|_{-r_0} < \infty\}.$$

Then $P(\Omega_1^T) = 1$ and if $\omega \in \Omega_1^T$ the real valued map $t \to \|M_t(\omega)\|_{-q_T}$ from $[0,T]$ to $\mathbb{R}$ is right continuous with left hand limits. Then by (14.5) in Billingsley (1968)

$$\sup_{0\le t\le T} \|M_t(\omega)\|_{-q_T} < \infty . \tag{3.2}$$

This fact enables us to show as in Step 1 of Theorem 1 that the map

$$\phi \to \int_0^t M_s[A_sT(s,t)\phi]ds$$

is linear and continuous on Φ.

As in Step 2 of Theorem 1 it follows that the putative solution (3.1) satisfies (V). We need only to replace W by M.

Next, as in Step 3 of Theorem 1 and using (3.2) it is easy to show that $Y_t(\omega)$ given by

$$Y_t(\omega)[\phi] = \int_0^t M_s[A_sT(s,t)\phi]ds \quad \text{for all } \phi \in \Phi \tag{3.3}$$

satisfies the inequality

$$|Y_t[\phi] - Y_{t_0}[\phi]| \le \sup_{0\le s\le T} \|M_s\|_{-q_T} TD\|\phi\|_r |t_0 - t| \tag{3.4}$$

for $t_0, t \in [0,T]$ and some $D > 0$, $r > 0$. Hence, $Y_t(\omega)[\phi]$ is continuous in t on

$[0,T]$ for each $\phi \in \Phi$ and $\omega \in \Omega_1^T$. Then by (3.1) $\xi_t(\omega)[\phi]$ is right continuous. The proof of the existence of a $D([0,T];\Phi'_{\ell_T})$-version is similar to the proof of Step 4 in Theorem 1 using again (3.2). The uniqueness is shown in a similar way.

Finally the semimartingale property of the solution is shown in a similar manner to Proposition 1.

Q.E.D.

Theorem 8. Assume A3(b)-(c), A4 and let $\eta = (\eta_t)_{t\geq 0}$ be a Φ'-valued right continuous stochastic process such that for each $T > 0$ there exists $q_T > 0$ such that

$$E(\sup_{0\leq t\leq T} \|\eta_t\|^2_{-q_T} < \infty) = 1.$$

Then the stochastic equation

$$\text{(VI)} \qquad \xi_t = \int_0^t T'(s,t)P'_s\xi'_s ds + \eta_t \qquad t \geq 0$$

has a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ such that for each $T > 0$ there exists $p_T > 0$ and

$$\xi^T_\cdot \in D([0,T];\Phi'_{p_T}) \quad \text{a.s.}$$

The proof is similar to that of Theorem 2. The only change is in Step 4 where we must show that $\xi_t[\phi]$ is right continuous and $\xi^T_\cdot \in D([0,T];\Phi'_{p_T})$ a.s.

Theorem 6 now follows from Theorems 7 and 8 using the same arguments as in the proof of Theorem 3.

4. Special Cases and Examples

In this section we consider special cases and examples of the above theorems.

Example 1. (Kallianpur and Wolpert (1984), Christensen (1985)).

Let $\Phi \hookrightarrow H \hookrightarrow \Phi'$ be a rigged Hilbert space on which is defined a continuous linear operator $A : \Phi \to \Phi$ and a strongly continuous semigroup $(T_t)_{t \geq 0}$ on the Hilbert space H such that the following conditions hold:

i) $T_t \Phi \subseteq \Phi \quad t \geq 0$.

ii) The restriction $T_t|_\Phi : \Phi \to \Phi$ is Φ-continuous for all $t \geq 0$.

iii) $t \to T_t \phi$ is continuous for all $\phi \in \Phi$.

iv) The generator $-L$ of T_t on H coincides with A on H.

A semigroup $(T_t)_{t \geq 0}$ satisfying the above conditions is called compatible with (Φ, H, Φ') or equivalently we say that (Φ, H, T_t) is a compatible family. If in addition we assume that some power $r_1 > 0$ of the resolvent $(\alpha I + L)^{-r_1}$ is a Hilbert-Schmidt operator, an appropriate countably Hilbertian nuclear space can be constructed in the following manner (see Kallianpur and Wolpert (1984) for details): The later condition on L implies that there is a CONS $\{\phi_j\}_{j \geq 1}$ in H such that $L\phi_j = \lambda_j \phi_j \quad j \geq 1$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. Take $\alpha = 1$ and define

$$\Phi = \{\phi \in H : \|(I + L)^r \phi\|_H^2 < \infty \text{ for all } r \in \mathbb{R}\}$$

$$= \{\phi \in H : \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, \phi_j \rangle_H^2 < \infty \text{ for all } r \in \mathbb{R}\}.$$

Define the inner product $\langle \cdot, \cdot \rangle_r$ on Φ by

$$\langle \phi, \psi \rangle_r := \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, \phi_j \rangle_H \langle \psi, \phi_j \rangle_H$$

and

$$\|\phi\|_r^2 = \langle \phi, \phi \rangle_r .$$

Let Φ_r be the $||\cdot||_r$-completion of Φ. We then have

$$\Phi = \bigcap_r \Phi_r, \quad \Phi' = \bigcap_r \Phi'_r$$

and for $r \le s$ $||\phi||_r \le ||\phi||_s$ and so $\Phi_s \subset \Phi_r$ with $\Phi_0 = H$. It can be shown that the canonical injection $\Phi_p \hookrightarrow \Phi_r$ is Hilbert-Schmidt for $p \ge r + r_1$ and that $\Phi \hookrightarrow H \hookrightarrow \Phi'$ is a rigged Hilbert space. A compatible family (Φ, H, T_t) constructed in this way is said to be <u>special</u>.

Consider the Ornstein-Uhlenbeck SDE

$$(4.1) \qquad \begin{cases} d\xi_t = A\xi_t dt + dM_t \\ \xi_0 = \gamma \end{cases}$$

This SDE has been solved by Kallianpur and Wolpert (1984) in the case of a special compatible family and M is a Φ'-valued process with independent increments (a Φ'-valued martingale) defined through a Poisson random measure, namely

$$(4.2) \qquad M_t[\phi] = \int_0^t \int_{\mathbb{R}\times X} a\phi(x)\tilde{N}(dadxds) \quad \phi \in \Phi$$

where $\tilde{N}(da,dx,dx)$ is a compensated Poisson random measure with variance $\mu(dadx)ds$ for some σ-finite μ on $\mathbb{R}\times X$. The last named authors showed that when M is as in (4.2) or a Φ'-valued Wiener process, both M_t and the solution of (4.1) belong to the space $D(\mathbb{R}_+;\Phi'_q)$ (or $C(\mathbb{R}_+;\Phi'_q)$ in the Wiener process case) where q is independent of t. Recently G. Kallianpur and S. Ramaswamy have given an example of a Φ'-valued Gaussian martingale X_t that does not satisfy the following condition: There exists p independent of t such that $X_t \in \Phi'_p$ for all $t \ge 0$ a.s. The example is as follows: Let (Φ, H, T_t) be a special compatible family with $\{\phi_j\}_{j\ge 1}$, $\{\lambda_j\}_{j\ge 1}$ and r_1 as above. Define for $\phi \in \Phi$

$$f(s,\phi) = \sum_{j=1}^{\infty} (1+\lambda_j)^s \langle\phi_j, \phi\rangle_H$$

and let $(B_s)_{s\geq 0}$ be a real-valued standard Brownian motion. For $t \geq 0$ and $\phi \in \Phi$ define

$$X_{t,\phi} = \int_0^t f(s,\phi)dB_s.$$

Then $X_{t,\phi}$ has a regularization $X_t[\phi]$ that is a Φ'-valued Gaussian martingale such that there does not exist $p > 0$ independent of t with $X_t \in \Phi'_p$ for all $t \geq 0$. Hence we cannot expect that Theorem 7 applied to $M = X$ will give a solution lying in $C(\mathbb{R}_+;\Phi'_p)$ for p independent of t.

In the case of a compatible family and when M_t is a Φ'-valued martingale, the SDE (4.1) has been solved by Christensen (1985).

The SDE (4.1) is a special case of the SDE (IV) in Section 3 where $A_t = A$ and $P_t = 0$ for all $t \geq 0$. Then we have the following result.

Theorem 9. Let (Φ,H,T_t) be a compatible family. Let γ be an F_0-measurable random variable such that $E||\gamma||^2_{-r_0} < \infty$ for some $r_0 > 0$ and $M = (M_t)_{t\geq 0}$ be a Φ'-valued right continuous martingale such that $E(M_t[\phi])^2 < \infty$ for all $\phi \in \Phi$. Then the SDE (4.1) has a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ given by

$$\xi_t[\phi] = \gamma[T_t\phi] + \int_0^t M_s[T_{t-s}A\phi]ds + M_t[\phi] \quad \text{for all } \phi \in \Phi.$$

Moreover ξ has the following property: For each $T > 0$ there exists $p_T > 0$ such that

$$\xi^T_{\cdot} \in D([0,T];\Phi'_{p_T}) \quad \text{a.s.}$$

and

$$E(\sup_{0\leq t\leq T} ||\xi_t||^2_{-p_T}) < \infty .$$

Proof. It follows from Theorem 7 since any compatible family (Φ,H,T_t) satisfies assumptions A1-A3 given in the introduction.

The SDE (4.1) is a model used in neurophysiological applications (see Kallianpur and Wolpert (1984) and Christensen and Kallianpur (1985)). However it is important to observe that in this field the kind of perturbations that occur are more likely to be nonlinear rather than linear. We hope to investigate such problems in future papers.

Example 2. (Adapted from Mitoma (1985)). This example is an instance where $T(s,t)$, A_t and P_t can all be defined directly on a countably Hilbert nuclear space Φ. It was recently considered by Mitoma (1985) in the case when Φ is obtained by modifying the space S. For the purpose of illustration we here consider S for which some simplifications are possible. Recall that the topology of S is given by the Hilbertian norms

$$\|\phi\|_n^2 = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{2n} |\phi^{(k)}(x)|^2 dx \quad n \geq 0, \tag{4.3}$$

and that this topology is also given by the family of seminorms

$$|||\phi|||_n = \sup_{0 \leq k \leq n} \sup_{x \in \mathbb{R}} (1+x^2)^n |\phi^{(k)}(x)| \quad n \geq 0. \tag{4.4}$$

For $\phi \in S$ and $t \geq 0$ define

$$(A_t\phi)(x) = \frac{1}{2}\alpha(x,t)^2\phi^{(2)}(x) + \beta(x,t)\phi^{(1)}(x) \tag{4.5}$$

where $\alpha(x,t)$ and $\beta(x,t)$ are uniformly bounded functions satisfying the following two properties:

(i) $D^k\alpha(x,t)$, $D^k\beta(x,t)$ $(D^k = \frac{d^k}{dx^k})$ are continuous and bounded in (x,t) for all $k \geq 0$,

(ii) $D^{(2)}\alpha(x,t)$ and $D^{(2)}\beta(x,t)$ are locally ε-Hölder continuous for some $0 < \varepsilon \leq 1$ and $\alpha(x,t)$, $\beta(x,t)$ are locally Lipschitz continuous in x.

Theorem 10. Let $\alpha(x,t)$, $\beta(x,t)$ as above and define A_t as in (4.5) Let P_t be any perturbation operator from S to S that satisfies assumption A4(a)-(b). Then the SDE

$$d\xi_t = (A'_t + P'_t)\xi_t dt + dW_t \quad \xi_0 = \gamma$$

has an S'-valued solution where W_t is an S'-valued Wiener process and γ is an S'-valued Gaussian random variable.

Proof. We have to prove that assumptions of Theorem 3 are satisfied.

Step 1. We first prove that A_t satisfies assumption A3(a). Since $\alpha(x,t)$ and $\beta(x,t)$ are C^∞ in x with bounded derivatives that are continuous in t, for each $T > 0$ there exist constants $C_i = C_i(T,n)$ $i = 1,\ldots, 3$ such that for $0 \le t \le T$

$$|(A_t\phi)^{(n)}(x)| \le C_1|\phi^{(n)}(x)| + C_2|\phi^{(n+1)}(x)| + C_3|\phi^{(n+2)}(x)| \tag{4.6}$$

and therefore from (4.5) for some constant $C(n,t) \ge 0$

$$||A_t\phi||_n^2 \le C(n,T)||\phi||_{n+2}^2 \quad \text{for all } \phi \in \Phi \quad 0 \le t \le T. \tag{4.7}$$

Hence, $A_t : S \to S$ is a continuous linear operator in the S-topology.

Next, since $\alpha(x,t)$ and $\beta(x,t)$ have derivatives in x bounded and continuous in (x,t), for all $k \ge 0$, $\phi \in \Phi$ and $x \in \mathbb{R}$ $(A_t\phi)^{(k)}(x)$ is continuous in t. Then using (4.6) and the dominated convergence theorem, from (4.3) we have that for all $n \ge 1$ and $\phi \in S$

$$||A_t\phi - A_s\phi||_n^2 = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{2n}|(A_t\phi)^k(x) - (A_s\phi)^k_{(x)}|^2 dx \underset{s \to t}{\to} 0$$

for $s,t \in [0,T]$. Then assumption A3(a) is satisfied.

Step 2. We check conditions A3(b)-(e) in Theorem 3. In order to do this we apply the ideas of Mitoma (1985) of using some results in Kunita (1982) but applying them to the space S (for which some simplifications are possible) instead of the nuclear space considered by Mitoma.

Let $B = (B(t))_{t\ge0}$ be a one dimensional Brownian motion and $\eta_{s,t}(x)$ be a unique solution of the Itô stochastic differential equation (see Condition (iii))

$$\eta_{s,t}(x) = x + \int_s^t \alpha(\eta_{s,r}(x),r)\,dB(r) + \int_s^t \beta(\eta_{s,r}(x),r)dr$$

$$\eta_{s,s}(x) = x \quad x \in \mathbb{R}.$$

For any $\phi \in S$ define

$$(T(s,t)\phi)(x) = E[\phi(\eta_{s,t}(x))] \tag{4.8}$$

(which is well defined since ϕ is bounded). From Kunita (1982) using (iii) we obtain Itô's forward and backward equations for $s<t$

$$\phi(\eta_{s,t}(x)) - \phi(x) = \int_s^t \alpha(\eta_{s,r}(x),r)\phi^{(1)}(\eta_{s,r}(x))dB(r) + \int_s^t (A_r\phi)(\eta_{s,r}(x))dr \tag{4.9}$$

$$\phi(\eta_{s,t}(x)) - \phi(x) = \int_s^t \alpha(x,r)D(\phi(\eta_{r,t}(x))d\hat{B}(r) + \int_s^t (A_r\phi\cdot\eta_{r,t})(x)dr \tag{4.10}$$

where the first term of (4.10) is the backward Itô integral and $(\phi\cdot\eta_{r,t})$ means composition.

Taking expected values in both sides of (4.9) we have

$$(T(s,t)\phi)(x) - \phi(x) = E(\int_s^t (A_r\phi)(\eta_{s,r}(x))dr). \tag{4.11}$$

But

$$(A_r\phi)(\eta_{s,r}(x)) = \frac{1}{2}\alpha(\eta_{s,r}(x),r)^2\phi^{(2)}(\eta_{s,r}(x)) + \beta(\eta_{s,r}(x,r))\cdot\phi^{(1)}(\eta_{s,r}(x)).$$

Then from the boundedness of $\alpha,\beta,\phi^{(1)}$ and $\phi^{(2)}$ and Fubini's theorem applied to (4.11) we obtain

$$(T(s,t)\phi)(x) - \phi(x) = \int_s^t E[A_r\phi(\eta_{s,r}(x))]dr = \int_s^t T(s,r)(A_r\phi)(x)dr.$$

Hence, we have the forward equation

$$\frac{d}{dt}T(s,t)\phi(x) = (T(s,t)A_t\phi)(x) \quad s<t,\ \phi \in S. \tag{4.12}$$

Similarly, taking expected values in both sides of (4.10) we have

$$(T(s,t)\phi)(x) - \phi(x) = E\int_s^t (A_r\phi\cdot\eta_{r,t})(x)dr$$

and since $A_r E(\phi(\eta_{r,t}(x)) = E((A_r\phi\cdot\eta_{r,t})(x))$, then

$$(T(s,t)\phi)(x) - \phi(x) = \int_s^t A_r T(r,t)\phi(x)dr$$

which gives the backward equation

$$\frac{d}{ds}T(s,t)\phi = -(A_s T(s,t)\phi)(x) \quad s<t,\ \phi\in S. \tag{4.13}$$

Hence A_t is the generator of a two parameter semigroup and satisfies assumption A3(b).

Next from Lemma 2.3 in Kunita (1982), for $n\geq 0$ and $s,t\in[0,T]$

$$E[(1+|\eta_{s,t}(x)|^2)^{-n}] \leq K(n,T)(1+x^2)^{-n}. \tag{4.14}$$

Hence for all $s,t\in[0,T]$ and $0\leq k\leq n$

$$E|\phi^{(k)}(\eta_{s,t}(x))|^2 = E\{\frac{(1+|\eta_{s,t}(x)|^2)^{n+1}}{(1+|\eta_{s,t}(x)|^2)^{n+1}}|\phi^{(k)}(\eta_{s,t}(x))|\}^2$$

$$\leq E(\frac{1}{(1+|\eta_{s,t}(x)|^2)^{2n+1}})E((1+|\eta_{s,t}(x)|^2)^{2(n+1)}|\phi^{(k)}(\eta_{s,t}(x))|^2$$

$$\leq K(n,T)\frac{1}{(1+x^2)^{2(n+1)}}|||\phi|||^2_{2(n+1)}$$

i.e.

$$E|\phi^{(k)}(\eta_{s,t}(x))|^2 \leq K(n,T)\frac{|||\phi|||^2_{2(n+1)}}{(1+x^2)^{2(n+1)}} \quad s,t\in[0,T],\ k=0,\dots,n. \tag{4.15}$$

Hence, using (4.15)

$$||T(s,t)\phi||^2_n = \sum_{k=0}^n \int_{\mathbb{R}}(1+x^2)^{2n}|D^{(k)}E\phi(\eta_{s,t}(x))|^2dx$$

$$\leq \sum_{k=0}^n \int_{\mathbb{R}} (1+x^2)^{2n}E|\phi^{(k)}(\eta_{s,t}(x))|^2dx \leq K(n,T)|||\phi|||^2_{2(n+1)} \sum_{k=0}^n \int_{\mathbb{R}} \frac{1}{(1+x^2)}dx < \infty$$

$$\therefore\ ||T(s,t)\phi||^2_n \leq C(n,T)|||\phi|||^2_{2(n+1)} \quad s,t\in[0,T].$$

Then from the last expression we have that $T(s,t)$ satisfies assumptions A3(c) and A3(f).

Next from Theorem 2.1 in Kunita (1982) $\eta_{s,t}(x)$ is continuous in (s,t,x). Then since ϕ has continuous derivatives, applying dominated convergence theorem twice together with (4.15) if $t \downarrow t_0$ and $0 \le s \le t_0 \le T$, $\phi \in S$ we have

$$\|T(s,t)\phi - T(s,t_0)\phi\|_n^2 = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{2n} |E(\phi^{(k)}(\eta_{s,t}(x)) - \phi^{(k)}(\eta_{s,t_0}(x))|^2 dx$$

$$\to 0.$$

Hence $T(s,t)$ satisfies A3(d) and similarly satisfies A3(e).

Moreover, using again (4.15) and a similar argument to that used in obtaining (4.15) we have

$$E|\phi^{(k)}(\eta_{s,t}(x))|^2 = E\{\frac{(1+|\eta_{s,t}(x)|^2)^n}{(1+|\eta_{s,t}(x)|^2)^n}|\phi^{(k)}(\eta_{s,t}(x))|\}^2$$

$$\le E(\frac{1}{(1+|\eta_{s,t}(x)|^2)^{2n}})E(1+|\eta_{s,t}(x)|^2)^{2n}|\phi^{(k)}(\eta_{s,t}(x)|^2)$$

$$\le K(n,t)\frac{|||\phi|||_n^2}{(1+x^2)^{2n}} \quad \text{for all } x \in \mathbb{R} \text{ and for all } s,t \in [0,T] \quad s < t.$$

Hence using (4.4) and the above inequality we have

$$|||T(s,t)\phi|||_n^2 = \sup_{0\le k\le n} \sup_{x\in\mathbb{R}} (1+x^2)^{2n} |E\phi^{(k)}(\eta_{s,t}(x))|^2$$

$$\le K(n,T)\,|||\phi|||_n^2 \quad n \ge 1, \quad s,t \in [0,T], \quad T > 0$$

and therefore $T(s,t)$ satisfies assumption A4(c) for the family of seminorms $\{|||\cdot|||_n;\ n \ge 0\}$ given by (4.4).

Then if $(P_t)_{t\ge0}$ is any perturbation operator on S that satisfies conditions A4(a)-(b), by Theorem 3 the SDE

$$d\xi_t = (A_t + P_t)'\xi_t dt + dW_t$$

$$\xi_0 = \gamma$$

has a unique S'-valued solution $\xi = (\xi_t)$ such that for each $T > 0$ there exists $m > 0$ and

$$\xi^T_\cdot \in C([0,T];S'_m) \quad \text{a.s.}$$

$$E(\sup_{0\le t\le T} ||\xi_s||^2_{-m}) < \infty. \qquad \text{Q.E.D.}$$

Example 3. (Hitsuda-Mitoma (1985), Mitoma (1985)). This example has been considered by Hitsuda and Mitoma (1985) and Mitoma (1985) in connection with central limit theorems for propagation of chaos (see McKean (1967)).

Let

$$\rho(x) = \begin{cases} c\cdot\exp(-1/(1-|x|^2)) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where c is such that $\int_{\mathbb{R}} \rho(x)dx = 1$. Let

$$\psi(x) = \int_{\mathbb{R}} e^{-|\mu|}\rho(x-\mu)d\mu$$

and $\theta(x) = 1/\psi(x)$. Let S be the space of rapidly decreasing functions and define

$$\Phi = \{\phi(x) = \theta(x)f(x) : f \in S\}. \tag{4.16}$$

For $\phi \in \Phi$ $(\phi(x) = \theta(x)f(x))$ define the following Hilbertian and non-Hilbertian seminorms on Φ

$$||\phi||_n^2 = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{2n} |\frac{d^k}{dx^k} f(x)|^2 dx \tag{4.17}$$

$$|||\phi|||_n = \sup_{0\leq k\leq n} \sup_{x\in\mathbb{R}} (1+x^2)^n |\frac{d^k}{dx^k} f(x)| \tag{4.18}$$

These norms define and equivalent Frechet topology on Φ and $\{\Phi, ||\cdot||_n \; n \geq 0\}$ is a countably Hilbertian nuclear space.

Next let a(x,y) and b(x,y) be bounded C^∞-functions in (x,y) and define

$$\alpha(x,t) = \int_{\mathbb{R}} a(x,y)\mu(dy,t) \tag{4.19}$$

$$\beta(x,t) = \int_{\mathbb{R}} b(x,y)\mu(dy,t) \tag{4.20}$$

where $\mu(dx,t)$ is the probability distribution of the solution X(t) of the real valued SDE

$$dX(t) = \alpha(X(t),t)dB_t + \beta(X(t),t)dt, \quad X_0 = \sigma \tag{4.21}$$

where B_t is a one dimensional Brownian motion and σ is a real valued r.v. independent of (B_t) such that $E(e^{c_0\sigma^2}) < \infty$ for some $c_0 > 0$. McKean (1967) has shown that the measure $\mu(t)$ has a density $\mu(x,t)$ and that $\alpha(x,t)$, $\beta(x,t)$ and $\mu(x,t)$ are C^∞-functions in $\mathbb{R}\times\mathbb{R}_+$.

<u>Theorem 11</u>. Let $a(x,y)$, $b(x,y)$, $\alpha(x,t)$ and $\beta(x,t)$ be as above and define for $\phi \in \Phi$ $(\phi(x) = \theta(x)f(x)\ \ f \in S)$ and $t \geq 0$

$$(4.22)\qquad (A_t\phi)(x) = \frac{1}{2}\alpha(x,t)^2\phi^{(2)}(x) + \beta(x,t)\phi^{(1)}(x)$$

$$(4.23)\qquad (P_t\phi)(x) = \int_{\mathbb{R}} b(y,x)\phi^{(1)}(y)\mu(dy,t) + \int_{\mathbb{R}} \alpha(y,t)a(y,x)\phi^{(2)}(y)\mu(dy,t).$$

Then the SDE

$$d\xi_t = (A_t + P_t)'\xi_t dt + dW_t \quad \xi_0 = \gamma$$

has a unique Φ'-valued solution, where W_t is a Φ'-valued Wiener process independent of the Φ'-valued Gaussian random variable γ.

<u>Proof</u>. We have to show that assumptions A1-A4 of Theorem 3 are satisfied. Conditions A1-A3 are shown in a similar way as in Example 2 (see Mitoma (1985)). It remains to show that the perturbation operator given by (4.23) satisfies assumptions A4(a)-(c).

Let $T > 0$ and for $0 \leq t \leq T$ define

$$g_t(x) = \int_{\mathbb{R}} b(y,x)\phi^{(1)}(y)\mu(dy,t)$$

and

$$h_t(x) = \int_{\mathbb{R}} a(y,x)\alpha(y,t)\phi^{(2)}(y)\mu(dy,t).$$

Then from (4.18) for $0 \leq t \leq T$ and $n \geq 0$

$$(4.24)\qquad |||g_t|||_n = \sup_{0\leq k\leq n}\ \sup_{x\in\mathbb{R}} (1+x^2)^n\left|\frac{d^k}{dx^k}\psi(x)g_t(x)\right|.$$

Using Leibnitz formula and the definition of $g_t(x)$ we have

$$\frac{d^k}{dx^k}\psi(x)g_t(x) = \sum_{j=0}^{k} \frac{d^k}{dx^k}\psi(x)\ \frac{d^{k-j}}{dx^{k-j}} g_t\ (x) = \sum_{j=0}^{k} \int_{\mathbb{R}} \frac{d^j}{dx^j}\psi(x)\frac{d^{k-j}}{dx^{k-j}} b(y,x)\phi^{(1)}(y)\mu(dy,t).$$

Next, using the fact that $b(x,y)$ is a uniformly bounded function in C^∞ we obtain that for a constant $K_1 = K_1(n)$

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \left|\frac{d^k}{dx^k}\psi(x)\ g_t(x)\right| \le K_1 \sum_{j=0}^{k} \sup_{x\in\mathbb{R}} (1+x^2)^n \left|\frac{d^j}{dx^j}\psi(x)\right| \int_{\mathbb{R}} |\phi^{(1)}(y)|\mu(dy,t).$$

But $\psi \in S$ since for each $n \ge 1$ $\left|\frac{d^n}{dx^n}\psi(x)\right| \le C(n)e^{-|x|}$. Thus

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \left|\frac{d^j}{dx^j}\psi(x)\right| < M_n \qquad j = 0,\dots, n$$

and hence

(4.25) $$|||g_t(x)|||_n \le K_2(n)\int_{\mathbb{R}} |\phi^{(1)}(y)|\mu(dy,t).$$

Next since $\phi(y) = \Theta(y)f(y)$,

$$|\phi^{(1)}(y)| \le |\Theta^{(1)}(y)||f(y)| + |\Theta(y)||f^{(1)}(y)|.$$

But for each $n \ge 0$

$$|\theta^{(n)}(y)| \le N_n e^{|y|} \qquad y \in \mathbb{R}$$

and by (4.18)

$$(1 + y^2)^2|f(y)| \le |||\phi|||_2 \qquad y \in \mathbb{R}$$

and

$$(1 + y^2)^2|f^{(1)}(y)| \le |||\phi|||_2 \qquad y \in \mathbb{R}.$$

Then

$$|\phi^{(1)}(y)| \le C_n e^{|y|}$$

and from (4.24) we have

$$|||g_t|||_n \le K_3(n)\,|||\phi|||_2 \int_{\mathbb{R}} e^{|y|} d\mu(t,dy).$$

Finally since $E[e^{C_0\sigma^2}] < \infty$ for some constant C_0, using Theorem 5.7.2 in Kallianpur (1980) we have

$$\int_{\mathbb{R}} e^{|y|} d\mu(t,dy) \le K_T \quad \text{for } 0 \le t \le T. \tag{4.26}$$

Then

$$|||g_t|||_n \le K_4(n)\, |||\phi|||_2 \quad 0 \le t \le T$$

and in a very similar way one shows that

$$|||h_t|||_n \le K_5(n)\, |||\phi|||_2 \quad 0 \le t \le T.$$

Hence for each $n \ge 1$ and $0 \le t \le T$

$$|||P_t\phi|||_n \le K_6(n,T)\, |||\phi|||_2 \tag{4.27}$$

and P_t satisfies assumption A4(a).

Next from (4.23) since a(x,y) and b(x,y) are uniformly bounded C^∞-functions in (x,y), from (4.26) we have that for $t, t_0 \in [0,T]$

$$\begin{aligned}
|(P_t\phi)^k(x) - (P_{t},\phi)^{(k)}(x)| &\le \Big|\int_{\mathbb{R}} \frac{\partial^k}{\partial x^k} b(y,x)\phi^{(1)}(y)(\mu(y,t) - \mu(y,t_0))dy\Big| \\
&\quad + \Big|\int_{\mathbb{R}} \frac{\partial^k}{\partial x^k} (y,x)\phi^{(2)}(y)(\alpha(y,t)\mu(t,y) - \alpha(y,t_0)\mu(y,t_0))dy\Big| \\
&\le K_5(k)\{ \int_{\mathbb{R}} e^y |\mu(y,t) - \mu(y,t_0)|dy \\
&\quad + \int_{\mathbb{R}} e^y |\alpha(y,t)\mu(y,t) - \alpha(y,t_0)\mu(y,t_0)|dy.
\end{aligned}$$

Also as in the proof of (4.25) we have

$$C_3 = \sup_{0\le k\le n} \sup_{x\in\mathbb{R}} (1 + x^2)^n \Big|\frac{d^k}{dx^k}\psi(x)\Big| < \infty .$$

Hence, using (4.18) and Leibnitz formula

$$|||(P_t\phi)^{(k)} - (P_{t_0}\phi)^{(k)}|||_n = \sup_{0\le k\le n} \sup_{x\in\mathbb{R}} (1+x^2)^n \Big|\frac{d^k}{dx^k}\psi(x)((P_t\phi)^{(k)}(x) - P_{t_0}\phi)^{(k)}(x)\Big|$$

$$\leq C_3 \sum_{j=0}^{k} K_5(j)\{ \int_{\mathbb{R}} e^y|\mu(y,t)-\mu(y,t_0)|dy + \int_{\mathbb{R}} e^y|\alpha(y,t)\mu(y,t)-\alpha(y,t_0)\mu(y,t_0)|dy\}$$

which goes to zero as $t \to t_0$, $t,t_0 \in [0,T]$ since $\alpha(y,t)$ and $\mu(t,y)$ are C^∞-functions in $\mathbb{R} \times \mathbb{R}_+$. Then P_t satisfies assumption A4(b) of Theorem 3. Q.E.D.

This example has been considered by Mitoma (1985) and Hitsuda and Mitoma (1985) in connection with the following central limit theorem: Consider the n-th interacting particle diffusion process $Y^{(n)}(t) = (Y_1^{(1)}(t),\ldots, Y_n^{(n)}(t))$ given by the SDE

$$Y_k^{(n)}(t) = \sigma_k + \frac{1}{n}\sum_{j=1}^{n}\int_0^t a(Y_k^{(n)}(s),Y_j^{(n)}(s))dB_k(s) + \frac{1}{n}\sum_{j=1}^{n}\int_0^t b(Y_k^{(n)}(s),Y_j^{(n)}(s))ds$$

$k=1,2,\ldots, n$,

where $(\sigma_k,B_k(t))_{k\geq 1}$ are independent copies of $(\sigma,B(t))$. Writing

$$U^{(n)}(t) = \frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k^{(n)}(t)} \qquad t>0$$

(where δ_x is the unit mass at x) McKean (1967) has shown that $U^{(n)}(t) \xrightarrow{a.s.} \mu(t)$ where $\mu(t)$ is the probability distribution of the solution of (4.31). Let

$$S_n(t) = \sqrt{n}(U^{(n)}(t)-\mu(t)).$$

Hitsuda and Mitoma (1985) have shown that any limit process $\xi = (\xi_t)$ of the measure valued process $S_n(\cdot)$ must satisfy the stochastic evolution equation

(4.28) $$d\xi_t = (A_t' + P_t')\xi_t dt + dW_t \qquad \xi_0 = \gamma$$

where A_t and P_t are given by (4.22) and (4.23), ξ_t is a Φ'-valued process and Φ is the countably Hilbert nuclear space given by (4.16).

Mitoma (1985) has solved the equation (4.28) under the additional hypotheses that all the derivatives with respect to x of $\alpha(x,t)$ and $\beta(x,t)$ are locally Holder $\lambda(n,t)$-continuous on T for each $n \geq 1$ and $T>0$. He considers (4.28) as

$$\xi_t = \gamma + W_t + \int_0^t (A_s + P_s)'\xi_s ds$$

where the integral means the Riemann integral and his proof requires the extension of Kolmogorov's forward and backward equation to the $|||\cdot|||_n$-completion of Φ for each $n \geq 1$.

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